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ON LINE SCREW SYSTEMS AND THEIR APPLICATION TO FLEXURE SYNTHESIS

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ABSTRACT

This paper presents a new screw theory that deals with “lines” and “line systems”. A line is a screw with a zero pitch. A rank m screw system is called a line screw system if it contains m linearly independent lines. The set of lines within a screw system is called the line variety. A general m -system is a line screw system if the rank of its line variety equals m . This paper will answer two questions: (1) how to calculate the rank of the line variety for a given m -system and (2) how to algorithmically find a set of linearly independent lines from the given screw system.

The motivation of this work comes from flexure synthesis whose goal is to find a pattern of flexures to achieve a specified motion. It has been previously found that a wire flexure is considered a line screw, or more specifically a pure force wrench. The synthesis of flexures is to remove undesired motion by applying constraints with flexures. By following reciprocity and definitions of line screws, we have derived the sufficient and necessary conditions of line screw systems. When applied to flexure synthesis, we show that not all motion pattern can be realized with wire flexures connected in parallel. A computational algorithm based on the line screw theory is developed to find all admissible line screws or force wrenches for a given motion space. At last two flexure synthesis case studies are provided to demonstrate the theory and the algorithm.

NOMENCLATURE

\mathbf{x} A row vector or a matrix in \mathbb{R}^3

\mathbf{I} The 3 by 3 identity matrix
 $\mathbf{0}$ The zero 3D row vector or zero matrix
 \mathbf{x}^T The transpose of a vector or a matrix
 $\hat{\$}$ A general screw which is a row vector in \mathbb{R}^6
 ${}^l\hat{\$}$ A line screw, also a row vector
 Π_S A screw matrix formed by independent screws
 \hat{T} A general twist which is a row vector in \mathbb{R}^6
 Π_T A twist matrix representing the motion space
 \hat{W} A general wrench which is a row vector in \mathbb{R}^6
 Π_W A wrench matrix representing the constraint space
 \hat{F} A force wrench which is a row vector in \mathbb{R}^6

1 Introduction

It is well known screw theory underlies the foundation of kinematics and statics. Even since it was first systematically studied by Ball [1], screw theory had been reinvented by many modern kinematician including Hunt [2, 3], Roth [4, 5], Waldron [6], Phillips [7, 8], Duffy and Lipkin [9, 10], Huang and Roth [11], Huang et al. [12], McCarthy [13, 14], Dai and Jones [15], Davidson and Hunt [16]. In the past decades, screw theory has been applied to a wide range of kinematic problems including constraint analysis [6], robotics [17, 18], assembly analysis [19, 20], mobility analysis [21], topology synthesis [22] and so on.

Here our interests are on lines and line varieties which are defined as the set of all lines within a screw system. The geometry of line varieties has been studied by H. Grassmann in 1800s and was introduced by Dandurand (1984) [23]. This geometric characterization is often called “Grassmann Geometry” has been

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applied to many kinematics problems. Merlet (1998) [24] studied singular configurations of parallel manipulators with Grassmann geometry. Hao and McCarthy [13, 14] systematically classified line-based singularities. Huang et al. (2008) [25] studied geometric characterization of finite screw systems with linear line complex which is a line variety of rank five.

Precision instruments and mechanisms [26] undergo a precise motion under the force or positional control. The central enabling elements of these precision mechanisms are flexures [27] which are typically thin beams or sheets and are designed to elastically deflect under external load. The flexure mechanisms are usually monolithic and considered as special types of compliant mechanisms [28]. One early step in the flexure design process is the synthesis of flexure pattern which seeks for an arrangement of flexures for a desired mobility of a functioning body or stage. The precision engineering community have been using the constraint-based approach [29–33] for flexure design. To achieve a given mobility pattern, the constraint-based method seeks for moving the undesired motion by applying constraints the functioning stage.

To analyze robot compliance, Pattern and Lipkin [34–36] systematically investigated the screw theory and its applications to compliance or elasticity analysis of robot manipulators. Huang and Schimmels [37–39] studied the realization of a prescribed stiffness matrix with simple and line springs. In compliant mechanism design, Kim (2008) [40] studied the characterization of compliant building blocks by utilizing the concept of eigentwists and eigenwrenches based on screw theory for designing compliant mechanisms. Su et al. (2009) [41] proposed a screw theory formulation of the constraint based design approach and showed its application to the analysis and synthesis of flexures. The fundamental premise of this approach is that a constraint and a motion correspond to a wrench and a twist in screw theory respectively. More specifically, the constraints applied by a wire or sheet flexure is formulated by a wrench system. If only using parallel connection patterns, they have shown that not all motion space can be achieved with simple flexures such as wires and sheets.

In this paper, we study a special kind of screw system called *line screw system*. A rank m screw system is called a line screw system if it contains m linearly independent lines. Essentially a general m -system is a line screw system if the rank of its line variety equals m . We derived the sufficient and necessary condition of a line screw system. To authors' knowledge, this is a new contribution to screw theory. We will also show how this line screw theory is applied to flexure synthesis. A computational synthesis algorithm is developed to determine the realizable motion spaces and find their reciprocal line screw system.

The rest of the paper is organized as follows. Section 2 introduces some basic concepts in screw theory. Section 3 defines lines, line varieties and line screw systems. Also in this section, we present two Lemmas regarding to the rank of line varieties.

Section 4 summarizes the criteria of line screw systems. Section 5 provides the synthesis procedure of applying the line screw theory to the flexure design. Section 6 provides two numerical examples of flexure design to demonstrate the theory and the algorithms. Section 7 presents conclusions.

2 Screws and Screw Systems

A screw is a geometric entity that either represents a general spatial motion of a rigid body or a wrench acting on the rigid body. Mathematically a general screw is denoted a six-dimensional row vector comprised of two row vectors $\mathbf{U}, \mathbf{V} \in \mathbb{R}^3$

$$\hat{\$} = (\mathbf{U} \mid \mathbf{V}) = (\mathbf{U} \mid \mathbf{p} \times \mathbf{U} + h\mathbf{U}) \quad (1)$$

where \mathbf{U} is a unit vector representing the direction of the screw axis, \mathbf{p} is a point on the screw axis, and the scalar h is the pitch.

2.1 The Screw Matrix

A rank m screw system or simply an m -system can be represented by m independent $\hat{\$}_1^T, \hat{\$}_2^T, \dots, \hat{\$}_m^T$ basis screws which we assemble into an m by 6 matrix called *screw matrix*,

$$\mathbf{\Pi}_S = \begin{bmatrix} \hat{\$}_1^T \\ \hat{\$}_2^T \\ \vdots \\ \hat{\$}_m^T \end{bmatrix} = [\mathbf{\Pi}_U \mid \mathbf{\Pi}_V] \quad (2)$$

where $\mathbf{\Pi}_U$ and $\mathbf{\Pi}_V$ are the left and right three columns of $\mathbf{\Pi}_S$ respectively. Any screw in the system can be written as a linear combination of the basis screws

$$\hat{\$} = \mathbf{a} \mathbf{\Pi}_S = \sum_{j=1}^m a_j \hat{\$}_j,$$

where $\mathbf{a} \in \mathbb{R}^m$ is an m dimensional row vector.

2.2 The Reciprocal Product

Two screws $\hat{\$}_1 = (\mathbf{U}_1 \mid \mathbf{V}_1)$, $\hat{\$}_2 = (\mathbf{U}_2 \mid \mathbf{V}_2)$ are called *reciprocal* if their reciprocal product is zero, i.e.

$$\hat{\$}_1 \circ \hat{\$}_2 = \hat{\$}_1 \Delta \hat{\$}_2^T = \mathbf{U}_1 \cdot \mathbf{V}_2 + \mathbf{U}_2 \cdot \mathbf{V}_1 = 0 \quad (3)$$

where the superscript “T” represents matrix transpose and

$$\Delta = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

is the column swap operator defined by Duffy and Lipkin (1985) [10].

Definition 1. The self-reciprocal product of a screw matrix $\Pi_{\mathcal{S}} = [\hat{\$}_1^T \hat{\$}_2^T \dots \hat{\$}_m^T]^T$ is its reciprocal product with itself, calculated as

$$\mathbf{Q}_{\mathcal{S}} = \Pi_{\mathcal{S}} \circ \Pi_{\mathcal{S}} = \Pi_{\mathcal{S}} \Delta \Pi_{\mathcal{S}}^T, \quad (4)$$

The product matrix $\mathbf{Q}_{\mathcal{S}}$ can be more specifically written as

$$\mathbf{Q}_{\mathcal{S}} = \Pi_U \Pi_V^T + \Pi_V \Pi_U^T = \begin{bmatrix} \hat{\$}_1 \circ \hat{\$}_1 & \hat{\$}_1 \circ \hat{\$}_2 & \dots & \hat{\$}_1 \circ \hat{\$}_m \\ \hat{\$}_2 \circ \hat{\$}_1 & \hat{\$}_2 \circ \hat{\$}_2 & \dots & \hat{\$}_2 \circ \hat{\$}_m \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\$}_m \circ \hat{\$}_1 & \hat{\$}_m \circ \hat{\$}_2 & \dots & \hat{\$}_m \circ \hat{\$}_m \end{bmatrix} \quad (5)$$

Note $\mathbf{Q}_{\mathcal{S}}$ is an m by m real symmetric matrix which can be always decomposed into

$$\mathbf{Q}_{\mathcal{S}} = \mathbf{P} \Lambda \mathbf{P}^T = \sum_{i=1}^m \lambda_i \mathbf{e}_i^T \mathbf{e}_i \quad (6)$$

where $\mathbf{P}\mathbf{P}^T = \mathbf{P}^T\mathbf{P} = \mathbf{I}$ and Λ is the diagonal matrix with eigenvalues λ_i on the diagonal.

3 Line Screw Systems

3.1 Lines and Line Varieties

In this paper, we are interested in line screws or simply lines which are screws with a zero pitch, i.e.

$${}^l\hat{\$} = (\mathbf{U} \mid \mathbf{V}): \quad \mathbf{U} \cdot \mathbf{V} = 0, \quad \mathbf{U} \neq \mathbf{0}, \quad (7)$$

where the superscript “ l ” indicates a line screw and “ \cdot ” is the dot product of two vectors. Physically a line screw can represent a pure force or a pure rotation.

Definition 2. Given an m -system represented by a screw matrix $\Pi_{\mathcal{S}} = [\hat{\$}_1^T \hat{\$}_2^T \dots \hat{\$}_m^T]^T$, we define the line variety as the set of all lines in the system, written as

$$\mathcal{L} = \{ {}^l\hat{\$} \mid {}^l\hat{\$} = (\mathbf{U} \mid \mathbf{V}) = \mathbf{a} \Pi_{\mathcal{S}}, \mathbf{U} \cdot \mathbf{V} = 0, \mathbf{U} \neq \mathbf{0}, \forall \mathbf{a} \in \mathbb{R}^m \}. \quad (8)$$

And the rank of \mathcal{L} is defined as the maximum number of linearly independent lines in the system, denoted by $\text{rank}(\mathcal{L})$.

Definition 3. An m -system is called a line screw system if the rank of its line variety \mathcal{L} equals m , i.e. $\text{rank}(\mathcal{L}) = m$.

Examples The following screw system is a line screw system,

$${}^l\Pi_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & 0 & \mid & 0 & 1 & 1 \\ 1 & -1 & 0 & \mid & 0 & 1 & -1 \end{bmatrix} \quad (9)$$

as we can find two linearly independent lines ${}^l\hat{\$}_1 = (1 \ 0 \ 0 \mid 0 \ 1 \ 0)$ and ${}^l\hat{\$}_2 = (0 \ 1 \ 0 \mid 0 \ 0 \ 1)$ which are the linear combinations of the two basis screws given above. Hence $\text{rank}(\mathcal{L}) = 2$.

However the following screw system is not a line screw system

$$\Pi_{\mathcal{S}} = \begin{bmatrix} 1 & 0 & 0 & \mid & 0 & 0 & 0 \\ 0 & 1 & 0 & \mid & 0 & 1 & 0 \end{bmatrix} \quad (10)$$

as the only line is ${}^l\hat{\$}_1 = (1 \ 0 \ 0 \mid 0 \ 0 \ 0)$. Hence $\text{rank}(\mathcal{L}) = 1$.

And $\text{rank}(\mathcal{L}) = 0$ for the following screw system

$$\Pi_{\mathcal{S}} = \begin{bmatrix} 0 & 0 & 0 & \mid & 1 & 0 & 0 \\ 0 & 1 & 0 & \mid & 0 & 1 & 0 \end{bmatrix} \quad (11)$$

as there is no lines in the system. All screws either have an infinite or a finite nonzero pitch.

Let us take a look at the properties of a line screw system. Given a general m -system $\Pi_{\mathcal{S}} = [\Pi_U \mid \Pi_V]$, let $\mu = \text{rank}(\Pi_U)$, $\nu = \text{rank}(\Pi_V)$. And let $\lambda_i (i = 1, \dots, m)$ be the eigenvalues of $\mathbf{Q}_{\mathcal{S}}$ and $k = \text{rank}(\mathbf{Q}_{\mathcal{S}})$ be the rank of $\mathbf{Q}_{\mathcal{S}}$ or the number of nonzero eigenvalues, where $\mathbf{Q}_{\mathcal{S}}$ is defined in (4).

Proposition 1. If $\Pi_{\mathcal{S}}$ is a line screw system, then $\mu \geq 1$ for $m = 1, 2, 3$, $\mu \geq 2$ for $m = 4, 5$ and $\mu = 3$ for $m = 6$.

Proof. Obviously for a line screw system, Π_U is an m by 3 nonzero matrix, hence $\mu \geq 1$ for $1 \leq m \leq 6$. And if $m = 6$, Π_U must have a full rank, hence $\mu = 3$.

If $m = 4, 5$, $\mu \geq 2$ is required since otherwise i.e. $\mu = 1$, all the line screws must be parallel. It is well known that at most three parallel lines in space are independent. Four or more parallel lines must be dependent. \square

Proposition 2. If $\Pi_{\mathcal{S}}$ is a line screw system, then its self-reciprocal product matrix $\mathbf{Q}_{\mathcal{S}}$ is indefinite (including the trivial case $\mathbf{Q}_{\mathcal{S}} = \mathbf{0}$), i.e. neither positive- nor negative-definite. Mathematically this is to say either $k = 0$ or eigenvalues λ_i have opposite signs when $k \geq 2$.

Proof. Since $\Pi_{\mathcal{S}}$ is a line screw system, there exist m independent lines denoted by

$${}^l\hat{\$}_j = \mathbf{a}_j \Pi_{\mathcal{S}} \quad j = 1, \dots, m. \quad (12)$$

which we assemble into the matrix form

$${}^l\Pi_S = \begin{bmatrix} {}^l\hat{s}_1 \\ {}^l\hat{s}_2 \\ \vdots \\ {}^l\hat{s}_m \end{bmatrix} = \mathbf{A} \Pi_S \quad (13)$$

where $\mathbf{A} = [\mathbf{a}_1^T, \dots, \mathbf{a}_m^T]^T$ is a full rank m by m matrix representing a coordinate transformation.

The self reciprocal product of the new screw matrix ${}^l\Pi_S$ can be written as

$${}^lQ_S = {}^l\Pi_S \Delta {}^l\Pi_S^T = \mathbf{A} Q_S \mathbf{A}^T. \quad (14)$$

which is a nonsingular congruence transformation of the original reciprocal product matrix Q_S .

Since all basis screws of ${}^l\Pi_S$ are lines, the diagonal entries of matrix lQ_S are zeros by observing (5). It follows that $tr({}^lQ_S) = \sum \lambda_i = 0$, where $tr(\cdot)$ is the trace of a matrix. This is possible only if either all eigenvalues are zeros ($k = 0$) or eigenvalues of lQ_S have opposite signs ($k \geq 2$). However by Sylvester's Law of Inertia, the number of positive, negative and zero eigenvalues is invariant under a nonsingular congruence transformation. It follows that eigenvalues of the original reciprocal product matrix Q_S must also be zeros or have opposite signs. This is equivalent to say that Q_S is indefinite. \square

Example The reciprocal product Q_S of the screw matrix given in (9) has eigenvalues are $(-2, 2)$ (indefinite) as it is a line screw system. While the screw system (10) cannot be a line screw system as its reciprocal product matrix Q_S has eigenvalues of $(2, 0)$ (positive semidefinite).

3.2 Finding the Rank of Line Varieties

Finding the line variety \mathcal{L} is equivalent to finding the following m dimensional space,

$$\{\mathbf{a} \in \mathbb{R}^m \mid \mathbf{a} Q_S \mathbf{a}^T = 0, \mathbf{a} \prod_U \neq \mathbf{0}\}, \quad (15)$$

where we have used the fact $\mathbf{U} = \mathbf{a} \prod_U$ and Eq. (4). Substituting (6) into (15) and defining $\mathbf{b} = \mathbf{a} \mathbf{P} = (b_1, \dots, b_m)$ yield

$$\mathcal{L}' = S \setminus \ker(\mathbf{P}^T \prod_U) = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} \Lambda \mathbf{b}^T = 0, \mathbf{b} \mathbf{P}^T \prod_U \neq \mathbf{0}\}, \quad (16)$$

where Λ is the diagonal matrix defined in (6), $\ker(\mathbf{P}^T \prod_U)$ is the kernel of matrix $\mathbf{P}^T \prod_U$, the symbol \setminus is the set minus operator. And $S \subset \mathbb{R}^m$ is an m dimensional linear space subject to a

quadratic constraint, defined by

$$S = \{\mathbf{b} = (b_1, \dots, b_m) \mid \mathbf{b} \Lambda \mathbf{b}^T = \sum_{i=1}^m \lambda_i b_i^2 = 0\}. \quad (17)$$

Obviously the spaces \mathcal{L} and \mathcal{L}' have the same rank, i.e. $rank(\mathcal{L}) = rank(\mathcal{L}') = rank(\mathcal{L}')$. Let $\mathbf{b}_j (j = 1, \dots, m)$ are a set of basis vectors of \mathcal{L}' . The basis of the \mathcal{L} can be uniquely calculated as

$${}^l\Pi_S = \begin{bmatrix} {}^l\hat{s}_1 \\ {}^l\hat{s}_2 \\ \vdots \\ {}^l\hat{s}_m \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix} \mathbf{P}^T \Pi_S.$$

In what follows, we will try to find $rank(\mathcal{L}')$. To do so, we first have to find $rank(S)$ which is defined as the number of nonzero linearly independent $\mathbf{b} \in S$ given in (17).

Lemma 1. *Let k be the number of nonzero eigenvalues of Q_S . And without the loss of generality, we assume $\lambda_1 \neq 0, \dots, \lambda_k \neq 0, \lambda_{k+1} = \dots = \lambda_m = 0$. The rank of S is given by*

$$rank(S) = \begin{cases} 0 & k \geq 2 \text{ and } \lambda_i \text{ have the same sign} \\ m-1 & k = 1 \\ m & k = 0 \text{ or } k \geq 2 \text{ and } \lambda_i \text{ have opposite signs} \end{cases} \quad (18)$$

Proof. 1. $k = 0$. This case is trivial. The quadratic constraint (17) does not apply any constraint on the linear system.

We arbitrarily choose m linearly independent $\mathbf{b}_j \in \mathbb{R}^m$, i.e. $rank(S) = m$.

2. $k = 1$. The quadratic constraint is $b_1 = 0$. The dimension of set S drops by one. We can find at most $m - 1$ linearly independent $\mathbf{b}_j = (0, b_{j2}, \dots, b_{jm})$, i.e. $rank(S) = m - 1$.

3. $k \geq 2$ and all nonzero coefficients $\lambda_1, \dots, \lambda_k$ have the same sign. The constraint equation $\sum_{i=1}^k \lambda_i b_i^2 = 0$ cannot have any real solution, i.e. $rank(S) = 0$.

4. $k \geq 2$ and $\lambda_1, \dots, \lambda_k$ have opposite signs. We discuss the following two cases.

(a) $k = 2$ The constraint equation is

$$\lambda_1 b_1^2 + \lambda_2 b_2^2 = 0, \lambda_1 \lambda_2 < 0 \implies$$

$$\mathbf{b}_{\pm} = (\pm b_2 \sqrt{-\lambda_2/\lambda_1}, b_2, b_3, \dots, b_m), b_2 \neq 0 \quad (19)$$

which represent two intersecting hyper-planes in \mathbb{R}^m . See Fig. 1(a). We can find m independent points on

these two planes as long as they are not chosen from the same hyperplane, i.e. $\text{rank}(\mathcal{S}) = m$.

- (b) $k > 2$ The constraint equation represents a quadratic surface in \mathbb{R}^m , from which we can always find m independent points, i.e. $\text{rank}(\mathcal{S}) = m$. See Fig. 1(b). \square

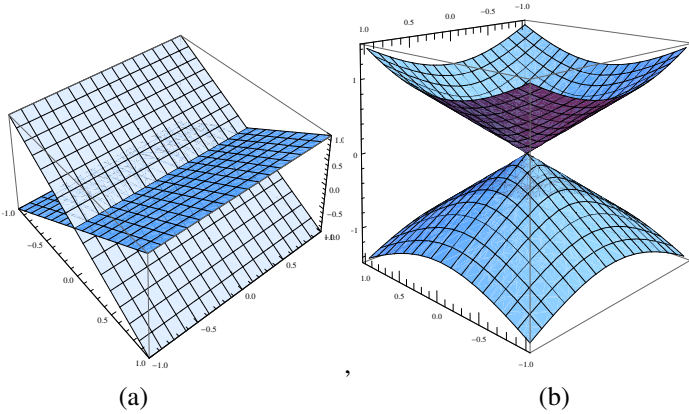


Figure 1. The quadratic constraint can be (a) two intersecting planes or (b) a cone surface

We now study $\text{rank}(\mathcal{L}')$. Obviously $\text{rank}(\mathcal{L}') \leq \text{rank}(\mathcal{S})$ as \mathcal{L}' is a subset of \mathcal{S} . Since we are only interested in the condition when $\text{rank}(\mathcal{L}') = m$, we only consider the case $\text{rank}(\mathcal{S}) = m$ shown in Lemma 1.

Lemma 2. *If $\text{rank}(\mathcal{S}) = m$, the rank of \mathcal{L}' is determined as the following:*

$$\text{rank}(\mathcal{L}') \begin{cases} = 0 & \mu = 0 \\ < m & \mu \geq 1, \mathbf{b}_+ \text{ or } \mathbf{b}_- \in \ker(\mathbf{P}^T \Pi_U) \text{ when } k = 2 \\ = m & \mu \geq 1, \mathbf{b}_\pm \notin \ker(\mathbf{P}^T \Pi_U) \text{ when } k = 2 \end{cases} \quad (20)$$

Proof. 1. $\mu = 0$ This is a trivial as the set \mathcal{L}' is null. Hence $\text{rank}(\mathcal{L}') = 0$. This makes sense because $\mu = 0$ implies $\Pi_U = \mathbf{0}$, i.e. all screws in the system must have an infinite pitch.

2. $\mu \geq 1$ Since Π_U is a nonzero m by 3 matrix and by linear algebra, the rank of $\ker(\mathbf{P}^T \Pi_U)$ is

$$\text{rank}(\ker(\mathbf{P}^T \Pi_U)) = \min(3, m) - \mu < m$$

This means that $\ker(\mathbf{P}^T \Pi_U)$ has a lower rank than \mathcal{S} . This leads to $\text{rank}(\mathcal{L}') = \text{rank}(\mathcal{S}) = m$ since in general if the rank is not changed if only a lower dimensional sub set is subtracted.

Geometrically this can be explained as follows. Generally, subtracting a finite number of surfaces from a volume still yields a volume space, subtracting a finite number of curves from a surface still yields a surface space and subtracting a finite number of points from a curve still yields a curve space. We call this case “a general case” which will be illustrated by Example 1 provided below.

However there is an exception which we call “the degenerated case”. If $k = 2$, the set \mathcal{S} are the two hyperplanes \mathbf{b}_\pm shown in (19) by Lemma 1. If one of the planes say $\mathbf{b}_+ \in \ker(\mathbf{P}^T \Pi_U)$, i.e. $\mathbf{b}_+ \mathbf{P}^T \Pi_U = \mathbf{0}$. We would have to pick all m points on the other plane \mathbf{b}_- . These m points can have at most $m - 1$ rank, i.e. $\text{rank}(\mathcal{L}') < m$. This case will be illustrated by Example 2 provided below. \square

We provide two examples to illustrate Lemma 2. Example 1 shows that subtracting $\ker(\mathbf{P}^T \Pi_U)$ from \mathcal{S} does not reduce its rank. While in example 2, the rank of \mathcal{S} is reduced due to the subtraction of $\ker(\mathbf{P}^T \Pi_U)$.

Example 1 Given a self-reciprocal screw system

$$\Pi_{\mathcal{S}} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right], \quad (21)$$

the eigenvalues of $Q_{\mathcal{S}}$ are $(2, -1, 1)$. By Lemma 1, the space \mathcal{S} is a quadratic surface $2b_1^2 - b_2^2 + b_3^2 = 0$ shown in Fig. 2(a). And the matrix $\mathbf{P}^T \Pi_U$ is

$$\mathbf{P}^T \Pi_U = \left[\begin{array}{cc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right]. \quad (22)$$

whose kernel is $\ker(\mathbf{P}^T \Pi_U) = (0, b_3, b_3)$. This represents a line through the origin shown as the red line in Figure 2(a). The space \mathcal{L}' is all the points on the quadratic surface \mathcal{S} excluding those on the line $(0, b_3, b_3)$. Obviously we can still find 3 independent in the space \mathcal{L} , i.e. $\text{rank}(\mathcal{L}') = 3$ as long as they are not chosen from the same line.

Example 2 Considering another screw system

$$\Pi_{\mathcal{S}} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right], \quad (23)$$

the eigenvalues of $Q_{\mathcal{S}}$ are $(-1, 1, 0)$. Solving the quadratic constraint equation yields $b_1 = \pm b_2$. By Lemma 1, the space \mathcal{S} is

comprised of two planes $\mathbf{b}_\pm = (b_2, \pm b_2, b_3)$. And the matrix $\mathbf{P}^T \Pi_U$ is

$$\mathbf{P}^T \Pi_U = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (24)$$

It has a $\ker(\mathbf{P}^T \Pi_U) = (b_2, b_2, b_3)$ which happens to be the plane \mathbf{b}_+ in \mathcal{S} . As a result, we can only find two independent points from the other plane \mathbf{b}_- , i.e. $\text{rank}(\mathcal{L}') = 2$. See Figure 2(b) for a geometric explanation of this case.

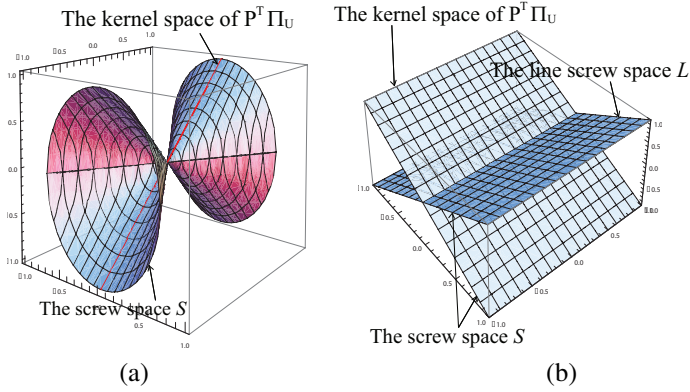


Figure 2. The parameterized line screw space \mathcal{L}

4 Criteria of Line Screw Systems

Now we come back to the question “given a screw system, how to determine whether or not it is a line screw system”. And if yes, how to algorithmically find all linearly independent lines.

We only need work on the cases $2 \leq m \leq 5$ as the cases $m = 1$ and $m = 6$ are trivial. If $m = 1$, the system has only one screw. If this screw is a line, the system is a line screw system, otherwise it is not. If $m = 6$, the system is always a line screw system since the screw matrix can be always row reduced into an identity matrix which gives the six principle screws. We can always find six line screws from the linear combinations of these six principle screws.

We will derive the sufficient and necessary condition or the criteria of line screw systems. Proposition 1 and 2 give us two necessary conditions of line screw systems. Combining them with Lemma 1 and 2 yields the sufficient and necessary condition of a rank m ($2 \leq m \leq 5$) line screw system which we summarize in the following Theorem.

Theorem 1. An m -system $\Pi_{\mathcal{S}} = [\Pi_U \mid \Pi_V]$ is a line screw system if and only if the following criteria are satisfied:

1. $\mu \geq 1$ for $m = 2, 3$, $\mu \geq 2$ for $m = 4, 5$,
2. the self-reciprocal product matrix $\mathbf{Q}_{\mathcal{S}}$ is indefinite,
3. $\mathbf{b}_\pm \notin \ker(\mathbf{P}^T \Pi_U)$ when $k = 2$. Here $\ker(\mathbf{P}^T \Pi_U)$ is the kernel of the matrix $\mathbf{P}^T \Pi_U$ and \mathbf{b}_\pm are defined in (19).

Proof. The necessity is obvious. To prove the sufficiency, we just need to prove $\text{rank}(\mathcal{L}) = \text{rank}(\mathcal{L}') = m$ if the above criteria are satisfied.

By Proposition 2, if $\mathbf{Q}_{\mathcal{S}}$ is indefinite, either all eigenvalue are zeros ($k = 0$) or eigenvalues ($k \geq 2$) have opposite signs. This further leads to $\text{rank}(\mathcal{S}) = m$ by Lemma 1. And by following Lemma 2, if $\mu \geq 1$ and $\mathbf{b}_\pm \notin \ker(\mathbf{P}^T \Pi_U)$ when $k = 2$, we have $\text{rank}(\mathcal{L}') = m$. Hence $\text{rank}(\mathcal{L}) = m$, i.e. the screw system $\Pi_{\mathcal{S}}$ is a line screw system. \square

4.1 Finding All Lines

If the criteria of line screw systems is satisfied, the procedure to find m independent lines is described as the following. All we have to do is to find m independent vectors \mathbf{b}_j ($j = 1, \dots, m$) from set \mathcal{L}' defined in (16). This is done by arbitrarily choosing m independent points from the set \mathcal{S} defined in (17).

If $k = 0$, \mathcal{S} is the whole \mathbb{R}^m . Any m independent points will be sufficient. If $k \geq 2$ and nonzero eigenvalues have opposite signs, let us assume $\lambda_1 \neq 0$. We solve the quadratic constraint equation in (17) for $b_1 = \pm \sqrt{(\sum_{i=2}^m \lambda_i b_i^2) / (-\lambda_1)}$. By appropriately sampling b_2, \dots, b_m such that the quantity under the square root is positive, we can always obtain m independent points. Note if $k = 2$, these m points must be chosen from both planes as shown in Lemma 1.

For all cases, we should check if the selected points are not within the set $\ker(\mathbf{P}^T \Pi_U)$. If yes, another set of m points must be chosen. This can be easily implemented using the Mathematica function “FindInstance” which exploits a random sampling procedure.

5 Synthesis of Flexures

Flexures are central structural elements of many precision instruments. The goal of flexure synthesis is to find a pattern of flexure arrangement to achieve a specified pattern of motion. Su et al. (2009) [41] proposed a screw theory based approach for the type synthesis of flexures in compliant mechanism design. In this framework, flexures are considered as constraints applied on a functional body and a twist system is used to characterize allowable motion of the body under the constraints. The constraint space given by a flexure arrangement (design) is mathematically denoted by a wrench system Π_W while the motion space is denoted by a twist system Π_T .

It is well known that for any given twist system $\Pi_T = [\hat{T}_1^T, \dots, \hat{T}_n^T]^T$ there exists a $m = 6 - n$ dimensional reciprocal

wrench system $\Pi_W = [\hat{W}_1^T, \dots, \hat{W}_m^T]^T$. Any wrench in Π_W is reciprocal to any twist in Π_T , i.e. $\hat{T}_i \circ \hat{W}_j = 0$. Given one system finding its reciprocal system is generally a linear process, see Dai and Jones (2003) [42] for example..

However in flexure design, we are particularly interested in finding the wrench system that consists of pure force wrenches which can be easily realized with wire flexures. According to the constraint based design approach [29], an ideal wire flexure eliminates the translational motion along the axial direction. Su et al. [41] have shown that the constraint applied by the ideal wire flexure is mathematically represented by a pure force wrench with the direction along the wire axis. As discussed before, this force wrench is essentially a line screw, written as

$$\hat{F} = (\mathbf{F} \mid \mathbf{M}) : \hat{F} \circ \hat{F} = 0, \quad \text{and} \quad \mathbf{F} \neq \mathbf{0}. \quad (25)$$

In what follows, we show how to apply the Theorems described above to flexure synthesis for compliant mechanism design.

5.1 The Realizable Motion Space

It has been demonstrated [41] that not all motion space can be realized by assembling wire flexures in parallel. In particular, we define the realizable motion space as follows.

Definition 4. A motion space represented by a twist matrix $\Pi_T = [\hat{T}_1^T, \dots, \hat{T}_n^T]^T$, is called realizable if there exist $m = 6 - n$ linearly independent pure force wrenches that are reciprocal to Π_T , i.e.

$$\Pi_W = \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \\ \vdots \\ \hat{F}_m \end{bmatrix} : \hat{T}_i \circ \hat{F}_j = 0, \hat{F}_j \circ \hat{F}_j = 0, \text{ and } \mathbf{F}_j \neq \mathbf{0} \quad (26)$$

where \mathbf{F}_j are the force component of \hat{F}_j , $i = 1, \dots, n$ and $j = 1, \dots, m$.

Please note that Definition 4 is different to the one given by Huang and Schimmels [37, 39]. Their focus is on the realization of a given stiffness matrix with simple or line springs. Their conclusion was that any full rank realizable stiffness matrix can be realized with a set of only line springs (finite stiffness) connected in parallel. While our goal is to determine if a given mobility pattern can be realized with ideal wire flexures which have an infinite stiffness along the flexure axis.

Following Theorem 1 and the definitions given previously, we have the following Corollary for determining if a n dimensional motion space is realizable.

Corollary 1. A motion space Π_T is realizable if and only if its reciprocal wrench system Π_W is a line screw system.

By checking the criteria listed in Theorem 1, we can determine if Π_W is a line screw system or not. If Π_W is a line screw system, we can find m linearly independent forces by following the procedure described in Section 4.1. We can then realize the design by designing a wire flexure along each of the m force wrenches.

5.2 The Synthesis Procedure

Figure 3 shows the flowchart of the synthesis procedure that returns the flexure pattern with any given motion pattern. The synthesis procedure starts with a given set of n twists that form a twist matrix Π_T and follows the steps below.

1. Compute the general reciprocal wrench matrix $\Pi_W = [\Pi_F \mid \Pi_M]$. This can be done by first computing the kernel matrix of Π_T and then swapping its first three columns with its last three columns. As an option, one can row reduce the matrix Π_W in order to obtain solutions along the coordinate axes.
2. Check the rank of the force component Π_F against the rank criteria given in Theorem 1. If the rank criteria is satisfied, continue the next step. Otherwise terminate the procedure.
3. Compute the self reciprocal product matrix $\mathbf{Q}_W = \Pi_W \circ \Pi_W$ and perform an eigenvalue decomposition,
4. Check if \mathbf{Q}_W is indefinite. If yes, continue to the next step. Otherwise terminate the procedure.
5. Check criteria 3 (degeneration condition) of Theorem 1 if $k = 2$. If it is satisfied, continue the next step. Otherwise terminate the procedure.
6. Follow the procedure described in Section 4.1 to choose $m = 6 - n$ independent vectors $\mathbf{b}_j = (b_{j1}, b_{j2}, \dots, b_{jm})$ that satisfy the following constraints

$$\sum_{i=1}^n \lambda_i b_{ji}^2 = 0 \quad \text{and} \quad \mathbf{b}_j \mathbf{P}^T \Pi_F \neq \mathbf{0}.$$

7. Return the following m independent self-reciprocal wrenches and terminate the procedure.

$$\Pi_W = \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \\ \vdots \\ \hat{F}_m \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix} \mathbf{P}^T \Pi_W.$$

If the given motion pattern is not realizable, the procedure returns failure flags that indicate the corresponding criteria not satisfied. The above synthesis procedure has been implemented in Mathematica software and is available upon request.

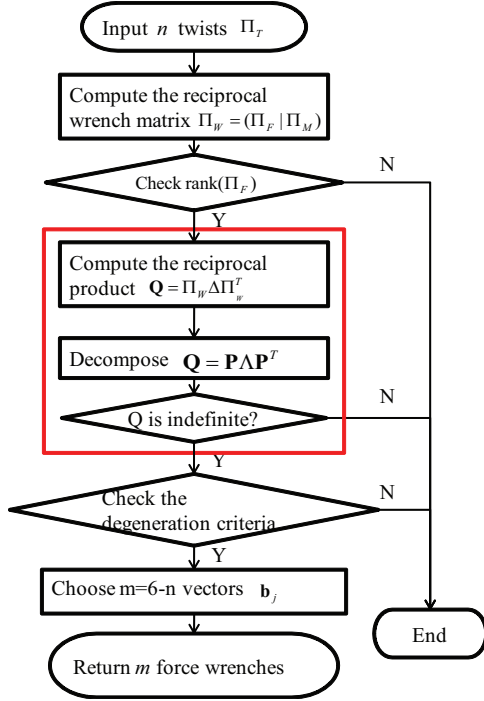


Figure 3. The flowchart of the flexure synthesis procedure

6 Case Studies

To test the algorithm, we have studied the synthesis of flexures for all combinations of orthogonal motions [43]. Furthermore, we provide two case studies in the following section.

Example 1

In this example, we seek for a constraint pattern design to achieve a two dof motion. One is the rotation about the x axis and the other is the rotation about the line through the point $(1, 0, 0)$ and parallel with the y axis. The twist matrix is

$$\Pi_T = \begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 0 & 0 \end{bmatrix}$$

Its reciprocal wrench system Π_W is calculated as

$$\Pi_W = \begin{bmatrix} 0 & 0 & 1 & | & 0 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 0 \\ 1 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

in which the first three rows are already force wrenches. However the last one is not. The rank of the force component of the matrix Π_W is 3, i.e. $\text{rank}(\Pi_F) = 3$. The eigenvalues of matrix \mathbf{Q}_W are $(-1, -1, 1, 1)$. By Theorem 1, we conclude that Π_W is a

line screw. By the following the synthesis procedure, we obtain the following four lines

$$\Pi_W = \begin{bmatrix} 0 & 0 & 1 & | & 0 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 0 \\ 1 & 0 & 0 & | & 0 & 1 & 0 \\ 1 & 0 & 0 & | & 0 & 1 & 1 \end{bmatrix}$$

Each wrench can be realized by a wire flexure. The flexure design is shown in Fig. 4(a) where the box represents the functioning body and the cylinders are wire flexures. Note the wire flexures are welded to the rigid body at one end and to the ground at the other end. One can easily check these four wires remove four degree-of-freedom of the body. Since all wrenches and twists are lines, each wrench line must be coplanar (intersect or parallel) with each twist lines. This agrees with our intuition and the well known results by the constraint-based design approach [29].

Example 2

In this example, the given twists are helical motions \hat{H}_x , \hat{H}_y and \hat{H}_z with pitch h_1 , h_2 and h_3 respectively, written as

$$\Pi_T = \begin{bmatrix} 1 & 0 & 0 & | & h_1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & h_2 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & h_3 \end{bmatrix}$$

The reciprocal wrench system Π_W is calculated as

$$\Pi_W = \begin{bmatrix} 1 & 0 & 0 & | & -h_1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & -h_2 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & -h_3 \end{bmatrix}$$

And the eigenvalues of the matrix \mathbf{Q}_W are $(-2h_1, -2h_2, -2h_3)$. Obviously the reciprocity of Π_W depends on the signs of the pitches. If all pitches are positive or negative, there does not exist any designs with wire flexures. Otherwise we can obtain valid designs with three wire flexures. For instance $h_1 = h_2 = 1, h_3 = -1$, we obtain the following three force wrenches

$$\Pi_F = \begin{bmatrix} 0 & 1 & 1 & | & 0 & -1 & 1 \\ -1 & 0 & 1 & | & 1 & 0 & 1 \\ -1 & 0 & -1 & | & 1 & 0 & -1 \end{bmatrix},$$

which correspond to the three flexures shown in Fig. 4(b). However this design is not intuitive and generally cannot be obtained by the the constraint-based design approach which clearly shows the benefits of the screw theory based approach.

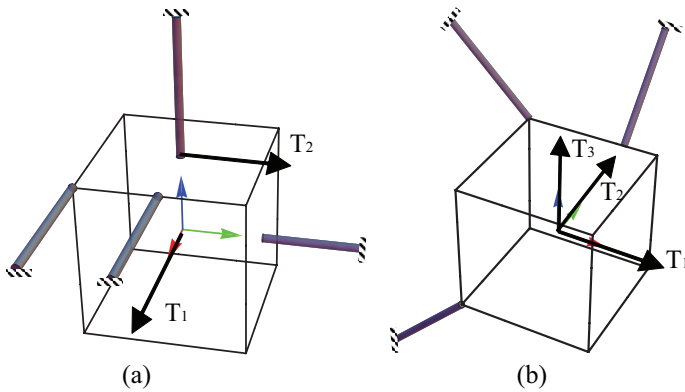


Figure 4. The flexure design for (a) Example 1 and (b) Example 2

7 Conclusion

In this paper, a formal definition of line screws and line screw systems are provided. The criteria of line screw systems are derived and demonstrated with numerical examples. The synthesis algorithm developed in the paper provides a systematical way to find flexure pattern designs for a given motion pattern. This line screw theory not only makes a new contribution to theoretical kinematics, but also lays out the foundation of the topological synthesis of flexures. It provides a guideline for designing flexures and justifies the constraint based design approach that heavily relies on designers' intuition and experience.

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