Proceedings of IDETC/CIE 2010
ASME 2010 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
August 15-18, 2010, Montreal, Quebec, Canada

DETC2010-28517

REALIZING ORTHOGONAL MOTIONS WITH WIRE FLEXURES CONNECTED IN PARALLEL

Hai-Jun Su,* Hafez Tari
Department of Mechanical Engineering
University of Maryland, Baltimore County
Baltimore, Maryland 21250
Email: haijun@umbc.edu, hafez.tari@umbc.edu

ABSTRACT

In this paper, we study the synthesis of wire flexures to achieve orthogonal motion by using a recently developed screw theory based approach. For a given desired mobility pattern, our goal is to find a system of wire flexures that are simply connected in parallel between the functional stage and the ground. It has been shown that a wire flexure is essentially a pure force or a line screw. An n dof motion space (allowable motion) is realizable if its reciprocal constraint space can be spanned by $6-n$ *line screws or forces. We first enumerate all possible one to five degree of motion spaces that are formed by motions along the coordinate axes attached on the functional stage. For each of these 34 motion spaces, we apply the screw theory approach to find its reciprocal force space as well as its rank. We conclude that 18 of them are realizable, 4 are realizable only when their pitches have opposite signs and 12 are not realizable. For each of these 34 cases, we provide an example showing the maximum number of independent wire flexures.*

1 Introduction

Flexures [1] are typically thin beams or sheets and are designed to elastically deflect under external loads. They are key functional motion elements for numerous precision instruments and mechanisms [2]. The flexure pivots [3] and mechanisms are typically monolithic and considered as special types of compliant mechanisms [4]. One important task of precision machine design is the synthesis of a flexure pattern which seeks for an arrangement of flexures for a desired mobility of a functioning body or stage. The constraint-based approach [5–7] has been widely used in precision engineering community flexure design.

Recently Hopkins and Culpepper (2009) [8, 9] have proposed a FACT method that systemizes the constraint design approach. In this approach, freedom and constraint topologies are represented by a set of geometric entities such as lines, pencil, hoop, plane, sphere etc. The major advantage of the FACT method is that it provides an intuitive visualization of the freedom or constraint space without advanced mathematical calculation. This geometric representation is very useful for simple cases. However these benefits are eclipsed for higher dimensional spaces as the visualization is too complicated for ordinary design engineers. For these complicated spaces, algebraic representation is preferable.

To overcome the difficulties of the constraint based approach, Su et. al (2009) [10] proposed a screw theory [11–18] formulation of the constraint based design approach and showed its application to the analysis and synthesis of flexures. The fundamental premise of this approach is that a constraint and a freedom correspond to a wrench and a twist respectively in screw theory. One main benefit of this approach is that it can be easily implemented with computer codes. They have demonstrated that not all freedom (or motion) space can be achieved with simple wire flexures connected in parallel. Recently Su and Tari (2010) [19] have derived the sufficient and necessary condition of the realizable motion space. They conclude that a motion space

*Address all correspondence to this author.

is realizable with wire flexures connected in parallel only if its reciprocal constraint space is a line screw system. A computational algorithm is developed to find all possible reciprocal forces in the constraint space for any given motion space.

In this paper, we focus on synthesizing simple connected flexure systems for given orthogonal motions along the coordinate axes of the functional stage. We first enumerate all possible $1 \leq n \leq 5$ degree-of-freedom spaces that allow motions along the coordinate axes attached on the functional stage. For each motion space, we apply the screw theory approach to find the reciprocal force space as well as its rank. We can then determine if the motion space is realizable or not.

The rest of the paper is organized as follows. Section 2 defines the line screw and line screw systems. Also in this section, the criteria of line screw system is provided. Section 3 provides the general synthesis procedure of applying line screw theory to the flexure design. Section 4 studies the synthesis of all possible orthogonal motion space. Section 5 presents conclusions.

2 Line Screws and Line Screw Systems

A general $\hat{\$} = (\mathbf{U} \mid \mathbf{V}) \in \mathbb{R}^6$ is considered a six dimensional row vector that is comprised of two three dimensional row vectors $\mathbf{U} \in \mathbb{R}^3$ and $\mathbf{V} \in \mathbb{R}^3$.

2.1 Definitions

In this paper, we are particularly interested in line screws and line screw systems. In general, a line screw could be a force wrench or a rotational twist. A formal definition is given below.

Definition 1. *A general screw $\hat{\$} = (\mathbf{U} \mid \mathbf{V})$ is called a line screw or a line for simplicity if it can be written as $\hat{\$} = (\mathbf{U} \mid \mathbf{p} \times \mathbf{U})$, where $\mathbf{U}$ represent the direction of the line and $\mathbf{p}$ represents a point on the line. In other words, a general line screw is subject to the following constraints*

$$^l\hat{\$} = (\mathbf{U} \quad \mid \quad \mathbf{V}): \quad \mathbf{U} \cdot \mathbf{V} = 0, \quad \mathbf{U} \neq \mathbf{0}, \tag{1}$$

where the superscript "l" indicates a line screw and "$\cdot$" is the dot product of two vectors.

A general rank m screw system or simply m-system can be spanned by m independent screws, denoted by a full rank m by 6 matrix, called *screw matrix* $\prod_{\$} = [\hat{\$}_1^T \; \hat{\$}_2^T \; \ldots \; \hat{\$}_m^T]^T$. Any m linearly independent screws in the system form the basis of the linear space. Any screw in the system can be written as a linear combination of the basis screws.

Let us define the line variety $\mathcal{L}$ as the set of all the line screws in an m-system $\prod_{\$}$, written as

$$\mathcal{L} = \left\{ {}^l\hat{\$} \mid {}^l\hat{\$} = (\mathbf{U} \mid \mathbf{V}) = \mathbf{a} \prod\nolimits_{\$}, \mathbf{U} \cdot \mathbf{V} = 0, \, \mathbf{U} \neq \mathbf{0}, \forall \mathbf{a} \in \mathbb{R}^m \right\}. \tag{2}$$

We also define the rank of $\mathcal{L}$ as the maximum number of linearly independent lines in the system, denoted by $rank(\mathcal{L})$. Obviously we have $rank(\mathcal{L}) \leq m$.

Definition 2. *An m-system $\prod_{\$}$ is called a line screw system if it contains of m linearly independent line screws, i.e. $rank(\mathcal{L}) = m$, where $\mathcal{L}$ is the line space of $\prod_{\$}$ defined in (2).*

2.2 Criteria of line screw systems

Let $\prod_{\$} = [\hat{\$}_1^T \; \hat{\$}_2^T \; \ldots \; \hat{\$}_m^T]^T = [\prod_U \mid \prod_V]$ denote an m-system, where $\prod_U$ and $\prod_V$ are left three columns and the right three columns of matrix $\prod_{\$}$ respectively.

We define the self-reciprocal product of a screw matrix $\prod_{\$}$ is its reciprocal product with itself, calculated as

$$\mathbf{Q}_{\$} = \prod\nolimits_{\$} \Delta \prod\nolimits_{\$}^T = \prod\nolimits_U \prod\nolimits_V^T + \prod\nolimits_V \prod\nolimits_U^T, \tag{3}$$

where the superscript "T" represents matrix transpose and

$$\Delta = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

is known as the column swap operator defined in [14].

Let $\lambda_i \in \mathbb{R}$ be the eigenvalues of $\mathbf{Q}_{\$}$. Without the loss of generality, we assume $\lambda_1 \neq 0, \ldots, \lambda_k \neq 0, \lambda_{k+1} = \cdots \lambda_m = 0$, where k is the number of nonzero eigenvalues. And let $\mathbf{P}$ be the matrix formed by the eigenvectors of $\mathbf{Q}_{\$}$. Su and Tari (2010) [19] have derived the sufficient and necessary conditions of line screw systems which are provided below for convenience.

Theorem 1. *$\prod_{\$}$ is a line screw system if and only if the following criteria are satisfied*

1. *$rank(\prod_U) \geq 1$ for $1 \leq m \leq 3$, $rank(\prod_U) \geq 2$ for $m = 4, 5$*
2. *$k = 0$ or $\lambda_1, \ldots, \lambda_k (k \geq 2)$ have opposite signs*
3. *$\mathbf{b}_{\pm} \notin ker(\mathbf{P}^T \prod_U)$ if $k = 2$, where $ker(\mathbf{P}^T \prod_U)$ defines the kernel of the matrix $\mathbf{P}^T \prod_U$ and $\mathbf{b}_{\pm}$ are*

$$\mathbf{b}_{\pm} = (\pm b_2 \sqrt{-\lambda_2/\lambda_1}, b_2, b_3, \ldots, b_m), b_2 \neq 0. \tag{4}$$

The detailed proof is given in Su and Tari (2010) [19].

3 Synthesis of Flexures

Flexures are central structural elements of many precision instruments. The goal of flexure synthesis is to find a pattern of flexure arrangement to achieve a specified pattern of motion.

3.1 Motion Space and Constraint Space

Su et al. (2009) [10] proposed a screw theory based approach for the type synthesis of flexures in compliant mechanism design. In this framework, flexures are considered as constraints applied on a functional body and a twist system is used to characterize allowable motion of the body under the constraints. The constraint space given by a flexure arrangement (design) is mathematically denoted by a wrench system $\prod_W$ while the motion space is denoted by a twist system $\prod_T$.

Definition 3. *A motion space is a screw system that is spanned by $n \leq 6$ linearly independent twists, represented by a twist matrix $\prod_T = [\hat{T}_1^T, \ldots, \hat{T}_n^T]^T$.*

A motion space essentially defines all possible allowable motion of a rigid body under a constraint pattern. Any allowable motion corresponds with a screw in the system which can be written as a linear combination of the basis twists $\hat{T}_i$.

Definition 4. *A constraint space is linear screw system that is spanned by $m \leq 6$ linearly independent wrenches, represented by a wrench matrix $\prod_W = [\hat{W}_1^T, \ldots, \hat{W}_m^T]^T$.*

The constraint space defines all formidable motion of a rigid body. Given one system finding its reciprocal system is generally a linear process, see Dai and Jones (2003) [20] for example. The problem of finding the reciprocal twist system for a given wrench system is called "flexure analysis", while the problem of finding the wrench system for a given twist system is called "flexure synthesis".

3.2 The Wire Flexure Wrench

An ideal wire flexure (Fig. 1(a)) is a slender structural member that is infinitely stiff along its axis but is infinitely compliant perpendicular to its axis. It is kinematically equivalent to a SS serial chain (Fig. 1(b)) where "S" represents a spherical joint or a ball-in-socket joint .

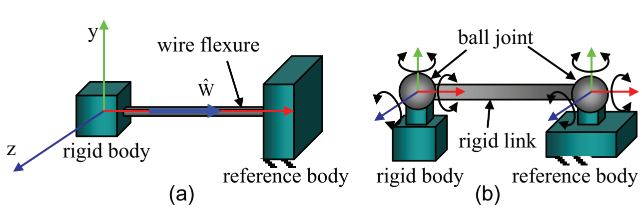

Figure 1. (a) an ideal wire flexure (b) a kinematically equivalent SS chain

According to the constraint-based design approach [5], an ideal wire flexure eliminates the translational freedom along the axial direction. Su et al. [10] have shown that the constraint applied by the ideal wire flexure is mathematically represented by a pure force wrench with the direction along the wire axis. As discussed before, this force wrench is essentially a line screw, written as

$$\hat{W} = (\mathbf{F} \mid \mathbf{M}) : \quad \hat{W} \circ \hat{W} = \mathbf{F} \cdot \mathbf{M} = 0, \text{ and } \mathbf{F} \neq \mathbf{0}, \tag{5}$$

where $\mathbf{F}$ represents the direction of the force and $\circ$ is known as the reciprocal product of two screws. If a wrench is actually a force, we denote it as $\hat{F}$.

3.3 Synthesis Procedure

The focus of this paper is on the synthesis of flexures which seeks for an arrangement of flexures for a given motion pattern. We are only interested in the simple parallel connection designs, i.e. the functioning body is supported by one or more wire flexures which are connected to a reference body in parallel as shown in Figure 2.

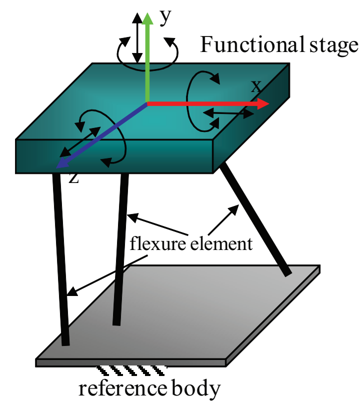

Figure 2. A functional stage is constrained by one or more wire flexures in parallel

Given a motion space denoted by the twist matrix $\prod_T = [\hat{T}_1^T, \ldots, \hat{T}_n^T]^T$, its reciprocal wrench space is written as

$$\mathcal{W} : \left\{ \hat{W} = (\mathbf{F} \mid \mathbf{M}) \mid \hat{T}_i \circ \hat{W} = 0, i = 1, \ldots, n \right\} \tag{6}$$

It is well known that for any given rank n twist system $\prod_T$, its reciprocal wrench space $\mathcal{W}$ is unique and has rank of $m = 6 - n$, i.e. $rank(\mathcal{W}) = 6 - n$. That is we can always find $m = 6 - n$ linearly independent wrenches $\hat{W}_j (j = 1, \ldots, m)$. However these wrenches $\hat{W}_j$ may be helical or couple wrenches which can not be easily realized with flexures.

When only force wrenches are considered, the reciprocal wrench system is given by

$$\mathcal{F} : \left\{ \hat{F} = (\mathbf{F} \mid \mathbf{M}) \mid \mathbf{F} \cdot \mathbf{M} = 0, \ \mathbf{F} \neq \mathbf{0}, \hat{T}_i \circ \hat{F} = 0, i = 1, \ldots, n \right\}. \tag{7}$$

Let $m = rank(\mathcal{F})$ be the maximum number of forces in $\mathcal{F}$. Let us assemble a set of m independent forces $\hat{F}_i \in \mathcal{F}$ into the matrix

$$^{l}\prod_W = [\hat{F}_1^T, \ldots, \hat{F}_m^T]^T = [\prod_F \mid \prod_M], \quad (8)$$

where the superscript "l" indicates a line screw system and $\prod_F$ and $\prod_M$ are the force and moment component of $\prod_W$ respectively.

And the flexure synthesis goal is to find $m = 6 - n$ forces in $\mathcal{F}$. However Su et. al [10] demonstrated that this is not always possible, i.e. $m < 6 - n$. Su and Tari [19] have shown that $m = 6 - n$ only if $\mathcal{F}$ is a line screw system. The sufficient and necessary condition of a line screw system is given in Theorem 1 and proved in [19]. In particular, if indeed $rank(\mathcal{F}) = 6 - n$, we call the motion space *realizable*.

Su and Tari [19] developed a computational algorithm for automatically finding up to $6-n$ forces for a given motion space. The returned forces are randomly chosen in the force space. However when the twist axes are parallel to the coordinate axes, we often would like to find designs with force axes parallel to the coordinate axes whenever possible. Note this often leads to intuitive designs. For this reason, the following synthesis procedure is preferred.

1. Write all desired motion in twists $\hat{T}_i (i = 1, \ldots, n)$
2. Substitute a general force $\hat{F} = (F_x\, F_y\, F_z \mid M_x\, M_y\, M_z)$ into the constraint equations (7).
3. Assign some of the components of $\hat{F}$ to zeros and solve the equation (7) for the rest of components.
4. Repeat step 3 until the maximum number of linearly independent forces are found.

4 Synthesis of Orthogonal Motions

In this section, we study the flexure synthesis for given motions with twist axes along the coordinate axes of the functioning stage. Hale (1999) [6] also studied the realization of orthogonal motions using the constraint based design approach. However the complete solution space is not obtained, and no helical motion is considered. Here we first enumerate all cases with one to five, $(1 \le n \le 5)$, degree-of-freedom motions. For each case, we derive the reciprocal force space $\mathcal{F}$ which consists of all the compatible solutions. For each case, we determined the dimension of $\mathcal{F}$. As shown before, the motion space is realizable if $rank(\mathcal{F}) = 6 - n$. We also provide an example design for each realizable motion space.

4.1 The orthogonal motion space

A rank n motion space is represented by a n linearly independent twists $\hat{T}_i$. To enumerate the cases when all twist axes are parallel the coordinate axes and through the origin. For instance, all the one, $(n = 1)$, dimensional motion space are

$$\begin{aligned}
\hat{R}_x &= \left(1\ 0\ 0 \mid 0\ 0\ 0\right) \\
\hat{R}_y &= \left(0\ 1\ 0 \mid 0\ 0\ 0\right) \\
\hat{R}_z &= \left(0\ 0\ 1 \mid 0\ 0\ 0\right) \\
\hat{P}_x &= \left(0\ 0\ 0 \mid 1\ 0\ 0\right) \\
\hat{P}_y &= \left(0\ 0\ 0 \mid 0\ 1\ 0\right) \\
\hat{P}_z &= \left(0\ 0\ 0 \mid 0\ 0\ 1\right) \\
\hat{H}_x &= \left(1\ 0\ 0 \mid p_x\ 0\ 0\right) \\
\hat{H}_y &= \left(0\ 1\ 0 \mid 0\ p_y\ 0\right) \\
\hat{H}_z &= \left(0\ 0\ 1 \mid 0\ 0\ p_z\right)
\end{aligned} \quad (9)$$

where $\hat{R}, \hat{P}, \hat{H}$ represent a rotational (R), prismatic (P) or helical (H) motion respectively and p_x, p_y, p_z are pitches.

And all two dimensional motion space with orthogonal axes are formed by any two distinct twists in (9). In particular, if two twist axes coincide, we name them as a cylindrical (C) motion, written as

$$\begin{aligned}
\hat{C}_x &= \begin{bmatrix} \hat{R}_x \\ \hat{P}_x \end{bmatrix} = \begin{bmatrix} 1\ 0\ 0 \mid 0\ 0\ 0 \\ 0\ 0\ 0 \mid 1\ 0\ 0 \end{bmatrix} \\
\hat{C}_y &= \begin{bmatrix} \hat{R}_y \\ \hat{P}_y \end{bmatrix} = \begin{bmatrix} 0\ 1\ 0 \mid 0\ 0\ 0 \\ 0\ 0\ 0 \mid 0\ 1\ 0 \end{bmatrix} \\
\hat{C}_z &= \begin{bmatrix} \hat{R}_z \\ \hat{P}_z \end{bmatrix} = \begin{bmatrix} 0\ 0\ 1 \mid 0\ 0\ 0 \\ 0\ 0\ 0 \mid 0\ 0\ 1 \end{bmatrix}
\end{aligned} \quad (10)$$

which is formed $\hat{R}$ and a $\hat{P}$ with the same twist axis. Note a $\hat{C}$ motion can be also generated by a $\hat{R}$ and $\hat{H}$ or by a $\hat{P}$ and a $\hat{H}$ as long as they have the same twist axis.

All the orthogonal motion space can be generated by using twists (9) and (10). In the following sections, we seek to find the flexure design for all possible orthogonal motion space with dimension $1 \le n \le 5$.

4.2 One degree-of-freedom motions

For the case of $n = 1$, the given twist can represent a rotational (R), prismatic (P) or helical (H) motion for which the goal is to find $m = 5$ linearly independent forces.

Without the loss of generality, we consider the helical motion along x axis and follow the synthesis procedure given in Section 3.3.

1. The only twist is $\hat{T}_1 = \hat{H}_x$ is given in (9).
2. Substitute $\hat{F} = (\mathbf{F} \mid \mathbf{M}) = (F_x\, F_y\, F_z \mid M_x\, M_y\, M_z)$ into (7) and

Table 1. Constraint spaces for one dof motions.

$\prod_T$	The reciprocal force space $\mathcal{F}$	$rank(\mathcal{F})$
$\hat{R}_x$	$\left(F_x\ F_y\ F_z \mid 0\ M_y\ M_z\right)$: $F_yM_y + F_zM_z = 0, F_x^2+F_y^2+F_z^2 \neq 0$	5
$\hat{P}_x$	$\left(0\ F_y\ F_z \mid M_x\ M_y\ M_z\right)$: $F_yM_y + F_zM_z = 0, F_y^2+F_z^2 \neq 0$	5
$\hat{H}_x$	$\left(F_x\ F_y\ F_z \mid -p_xF_x\ M_y\ M_z\right)$: $-p_xF_x^2 + F_yM_y + F_zM_z = 0, F_x^2+F_y^2+F_z^2 \neq 0$	5

obtain

$$\begin{cases} \hat{F} \circ \hat{H}_x = 0 \implies M_x + p_xF_x = 0 \implies M_x = -p_xF_x \\ \mathbf{F}\cdot\mathbf{M} = 0 \implies F_xM_x + F_yM_y + F_zM_z = 0 \\ \mathbf{F} \neq 0 \implies F_x^2+F_y^2+F_z^2 \neq 0 \end{cases},$$

The reciprocal force must have the following form

$$\left(F_x\ F_y\ F_z \mid -p_xF_x\ M_y\ M_z\right): \\ -p_xF_x^2 + F_yM_y + F_zM_z = 0, F_x^2+F_y^2+F_z^2 \neq 0$$

Any wrench stratifying the above conditions is a force reciprocal to the motion space.

3. Let $F_x = F_z = M_y = M_z = 0, F_y = 1$, we obtain one force $\left(0\ 1\ 0 \mid 0\ 0\ 0\right)$ which is realized with a wire flexure along the y axis.
4. Repeat step 3 until we obtain $m = 5$ forces, shown below

$$^l\prod_W = \begin{bmatrix} 0\ 1\ 0 \mid & 0 & 0 & 0 \\ 0\ 1\ 0 \mid & 1 & 0 & 0 \\ 0\ 0\ 1 \mid & 0 & 0 & 0 \\ 0\ 0\ 1 \mid & 1 & 0 & 0 \\ 1\ 0\ 1 \mid & -p_x & 0 & p_x \end{bmatrix}. \tag{11}$$

Please note that the results of steps 3 and 4 are not unique. This means that there are multiple flexure patterns satisfying the design goal.

Similarly we can obtain the reciprocal force space for one revolute motion and one prismatic motion. The results are tabulated in Table 1. An example design for each of these three cases is provided in Figure 3.

The conclusion is that any given one degree-of-freedom motion is realizable no matter if it is revolute, prismatic or helical. This result applies to the case when the twist axis is a general line in space as we can always define the coordinate x axis along the twist axis.

4.3 Two degree-of-freedom motions

For $n = 2$, two twists are given. Without the loss of generality, we assume the two twists are along x and y axes and obtain seven synthesis cases $\hat{R}_x\hat{R}_y, \hat{C}_x, \hat{R}_x\hat{P}_y, \hat{R}_x\hat{H}_y, \hat{P}_x\hat{H}_y, \hat{H}_x\hat{H}_y, \hat{P}_x\hat{P}_y$, shown in Table 2.

The synthesis task is to find $m \leq 4$ linearly independent forces that are reciprocal to $\prod_T$. As an example, we consider the twist matrix $\prod_T = [\hat{H}_x^T, \hat{H}_y^T]^T$, written as

$$\prod_T = \begin{bmatrix} 1\ 0\ 0 \mid p_x & 0 & 0 \\ 0\ 1\ 0 \mid 0 & p_y & 0 \end{bmatrix}.$$

Following the synthesis steps, we obtain the parameterized reciprocal force space

$$\left(F_x\ F_y\ F_z \mid -p_xF_x\ -p_yM_y\ M_z\right): \\ -p_xF_x^2 - p_yF_y^2 + F_zM_z = 0, F_x^2+F_y^2+F_z^2 \neq 0.$$

A design in the above solution space is shown below

$$^l\prod_W = \begin{bmatrix} 1 & 0 & 1 & -p_x & 0 & p_x \\ 1 & 0 & -1 & -p_x & 0 & -p_x \\ 0 & 1 & 1 & 0 & -p_y & p_y \\ 0 & 1 & -1 & 0 & -p_y & -p_y \end{bmatrix}.$$

Hence we conclude that the motion space formed by two orthogonal and intersecting helical twists is always realizable. Likewise we can draw the same conclusion for the cases $\hat{R}_x\hat{R}_y, \hat{C}_x, \hat{R}_x\hat{P}_y, \hat{R}_x\hat{H}_y, \hat{P}_x\hat{H}_y$.

However this conclusion does not hold for the case $\hat{P}_x\hat{P}_y$, in which the twist matrix is given by

$$\prod_T = \begin{bmatrix} 0\ 0\ 0 \mid 1\ 0\ 0 \\ 0\ 0\ 0 \mid 0\ 1\ 0 \end{bmatrix}. \tag{12}$$

Its reciprocal force space $\mathcal{F}$ is given by

$$\left(0\ 0\ F_z \mid M_x\ M_y\ 0\right): F_z \neq 0.$$

Obviously the $rank(\mathcal{F}) = 3$, i.e. only three forces can be found in $\mathcal{F}$. An example set of three independent wire flexures for two prismatic motions is given below

$$^l\prod_W = \begin{bmatrix} 0\ 0\ 1 \mid 0\ 0\ 0 \\ 0\ 0\ 1 \mid 1\ 0\ 0 \\ 0\ 0\ 1 \mid 0\ 1\ 0 \end{bmatrix}$$

Table 2. Constraint spaces for two dof motions.

$\prod_T$	The reciprocal force space $\mathcal{F}$	$rank(\mathcal{F})$
$\hat{R}_x\hat{R}_y$	$\left(F_x\ F_y\ F_z \mid 0\ 0\ M_z\right)$:	4
	$F_zM_z = 0, F_x^2+F_y^2+F_z^2 \neq 0$	
$\hat{C}_x$	$\left(0\ F_y\ F_z \mid 0\ M_y\ M_z\right)$:	4
	$F_yM_y + F_zM_z = 0, F_y^2+F_z^2 \neq 0$	
$\hat{R}_x\hat{P}_y$	$\left(F_x\ 0\ F_z \mid 0\ M_y\ M_z\right)$:	4
	$F_zM_z = 0, F_x^2+F_z^2 \neq 0$	
$\hat{R}_x\hat{H}_y$	$\left(F_x\ F_y\ F_z \mid 0\ -p_yF_y\ M_z\right)$:	4
	$-p_yF_y^2 + F_zM_z = 0, F_x^2+F_y^2+F_z^2 \neq 0$	
$\hat{P}_x\hat{H}_y$	$\left(0\ F_y\ F_z \mid M_x\ -p_yF_y\ M_z\right)$:	4
	$-p_yF_y^2 + F_zM_z = 0, F_y^2+F_z^2 \neq 0$	
$\hat{H}_x\hat{H}_y$	$\left(F_x\ F_y\ F_z \mid -p_xF_x\ -p_yM_y\ M_z\right)$:	4
	$-p_xF_x^2 - p_yF_y^2 + F_zM_z = 0, F_x^2+F_y^2+F_z^2 \neq 0$	
$\hat{P}_x\hat{P}_y$	$\left(0\ 0\ F_z \mid M_x\ M_y\ 0\right) : F_z \neq 0$	3

which are shown in Figure 5.

An alternative way to consider this case is provided below. The reciprocal wrench system $\prod_W$ of $\prod_T$ can be obtained by a linear process and can be row reduced into

$$\prod_W = \begin{bmatrix} 0\ 0\ 1 \mid 0\ 0\ 0 \\ 0\ 0\ 0 \mid 1\ 0\ 0 \\ 0\ 0\ 0 \mid 0\ 1\ 0 \\ 0\ 0\ 0 \mid 0\ 0\ 1 \end{bmatrix}.$$

If $\prod_W$ is a line screw system, the motion space $\prod_T$ is realizable. Let $\prod_F$ and $\prod_M$ be its force component and moment component of $\prod_W$ respectively. Obviously we can see that $rank(\prod_F) = 1$ which clearly does not satisfy the rank criteria of Theorem 1. Therefore $\prod_T$ in (12) is not realizable.

The reciprocal force spaces for two dof motions and their rank are tabulated in Table 2. Example designs for each of the six realizable motions space are given in Figure 3. The conclusion is that a two degree-of-freedom motion with orthogonal and intersecting twist axes is realizable except the case when both motions are translational.

4.4 Three degree-of-freedom motions

For $n = 3$ given twists, we are looking for $m = 3$ linearly independent forces. There are total of 14 cases shown in Table 3. Six of them, $\hat{R}_x\hat{R}_y\hat{R}_z$,$\hat{C}_x\hat{R}_y$,$\hat{R}_x\hat{R}_y\hat{P}_z$,$\hat{R}_x\hat{P}_y\hat{P}_z$,$\hat{C}_x\hat{H}_y$,$\hat{R}_x\hat{P}_y\hat{H}_y$ are always realizable. Example designs are given in Figure 3.

For motions with two or more helical dof, i.e. $\hat{R}_x\hat{H}_y\hat{H}_z$, $\hat{P}_x\hat{H}_y\hat{H}_z$ and $\hat{H}_x\hat{H}_y\hat{H}_z$, their realizability depends on the signs of the pitches. These cases are shown in Figure 4.

Table 3. Constraint spaces for three dof motions.

$\prod_T$	The reciprocal force space $\mathcal{F}$	$rank(\mathcal{F})$
$\hat{R}_x\hat{R}_y\hat{R}_z$	$\left(F_x\ F_y\ F_z \mid 0\ 0\ 0\right) :, F_x^2+F_y^2+F_z^2 \neq 0$	3
$\hat{C}_x\hat{R}_y$	$\left(0\ F_y\ F_z \mid 0\ 0\ M_z\right) : F_zM_z = 0, F_y^2+F_z^2 \neq 0$	3
$\hat{R}_x\hat{R}_y\hat{P}_z$	$\left(F_x\ F_y\ 0 \mid 0\ 0\ M_z\right) : F_x^2+F_y^2 \neq 0$	3
$\hat{R}_x\hat{P}_y\hat{P}_z$	$\left(F_x\ 0\ 0 \mid 0\ M_y\ M_z\right) : F_x \neq 0$	3
$\hat{C}_x\hat{H}_y$	$\left(0\ F_y\ F_z \mid 0\ -p_yF_y\ M_z\right)$:	3
	$-p_yF_y^2 + F_zM_z = 0, F_y^2+F_z^2 \neq 0$	
$\hat{R}_x\hat{P}_y\hat{H}_y$	$\left(F_x\ 0\ F_z \mid 0\ 0\ M_z\right) : F_zM_z = 0, F_x^2+F_z^2 \neq 0$	3
$\hat{R}_x\hat{H}_y\hat{H}_z$	$\left(F_x\ F_y\ F_z \mid 0\ -p_yF_y\ -p_zF_z\right)$:	3/0*
	$p_yF_y^2 + p_zF_z^2 = 0, F_x^2+F_y^2+F_z^2 \neq 0$	
$\hat{P}_x\hat{H}_y\hat{H}_z$	$\left(0\ F_y\ F_z \mid M_x\ -p_yF_y\ -p_zF_z\right)$:	3/0*
	$p_yF_y^2 + p_zF_z^2 = 0, F_y^2+F_z^2 \neq 0$	
$\hat{H}_x\hat{H}_y\hat{H}_z$	$\left(F_x\ F_y\ F_z \mid -p_xF_x\ -p_yF_y\ -p_zF_z\right)$:	3/0*
	$p_xF_x^2 + p_yF_y^2 + p_zF_z^2 = 0, F_x^2+F_y^2+F_z^2 \neq 0$	
$\hat{C}_x\hat{P}_y$	$\left(0\ 0\ F_z \mid 0\ M_y\ 0\right) : F_z \neq 0$	2
$\hat{R}_x\hat{R}_y\hat{H}_z$	$\left(F_x\ F_y\ 0 \mid 0\ 0\ 0\right) : F_x^2+F_y^2 \neq 0$	2
$\hat{R}_x\hat{P}_y\hat{H}_z$	$\left(F_x\ 0\ 0 \mid 0\ M_y\ 0\right) : F_x \neq 0$	2
$\hat{P}_x\hat{P}_y\hat{H}_z$	$\left(0\ 0\ 0 \mid M_x\ M_y\ 0\right)$	0
$\hat{P}_x\hat{P}_y\hat{P}_z$	$\left(0\ 0\ 0 \mid M_x\ M_y\ M_z\right)$	0

*0 if pitches have the same sign, 3 otherwise.

As an example, we consider three helical motions $\hat{H}_x$, $\hat{H}_y$ and $\hat{H}_z$ with pitches p_x, p_y and p_z respectively. The twist matrix for this motion space is written as

$$\prod_T = \begin{bmatrix} 1\ 0\ 0 \mid p_x & 0 & 0 \\ 0\ 1\ 0 \mid 0 & p_y & 0 \\ 0\ 0\ 1 \mid 0 & 0 & p_z \end{bmatrix}.$$

The reciprocal constraint (force) space $\mathcal{F}$ is

$$\left(F_x\ F_y\ F_z \mid -p_xF_x\ -p_yF_y\ -p_zF_z\right) :$$
$$p_xF_x^2 + p_yF_y^2 + p_zF_z^2 = 0, \quad F_x^2+F_y^2+F_z^2 \neq 0.$$

Obviously $\mathcal{F}$ has forces only if pitches p_x, p_y, p_z have opposite signs. Or we can follow Theorem 1 and calculate the reciprocal

Table 4. Constraint spaces for four dof motions.

$\prod_T$	The reciprocal force space $\mathcal{F}$	$rank(\mathcal{F})$
$\hat{C}_x\hat{R}_y\hat{R}_z$	$\left(0\ F_y\ F_z \mid 0\ 0\ 0\right): F_y^2+F_z^2 \neq 0$	2
$\hat{C}_x\hat{R}_y\hat{P}_z$	$\left(0\ F_y\ 0 \mid 0\ 0\ M_z\right): F_y \neq 0$	2
$\hat{C}_x\hat{H}_y\hat{H}_z$	$\left(0\ F_y\ F_z \mid 0\ -p_yF_y\ -p_zF_z\right)$: $p_yF_y^2+p_zF_z^2=0, F_y^2+F_z^2 \neq 0$	2/0*
$\hat{C}_x\hat{R}_y\hat{P}_y$	$\left(0\ 0\ F_z \mid 0\ 0\ 0\right): F_z \neq 0$	1
$\hat{C}_x\hat{R}_y\hat{H}_z$	$\left(0\ F_y\ 0 \mid 0\ 0\ 0\right): F_y \neq 0$	1
$\hat{C}_x\hat{P}_y\hat{P}_z$	$\left(0\ 0\ 0 \mid 0\ M_y\ M_z\right)$:	0
$\hat{C}_x\hat{P}_y\hat{H}_z$	$\left(0\ 0\ 0 \mid 0\ M_y\ -p_zF_z\right)$	0

*0 if pitches have the same sign, 2 otherwise.

wrench system $\prod_W$ of the twist matrix as

$$\prod_W = \begin{bmatrix} 1\ 0\ 0 \mid -p_x & 0 & 0 \\ 0\ 1\ 0 \mid \ \ 0 & -p_y & 0 \\ 0\ 0\ 1 \mid \ \ 0 & 0 & -p_z \end{bmatrix}.$$

The eigenvalues of the matrix $\mathbf{Q}_W = \prod_W \circ \prod_W$ are $(-2p_x, -2p_y, -2p_z)$. By Theorem 1, we conclude that the self-reciprocity of $\prod_W$ depends on the signs of the pitches. If all pitches are of the same sign, there will not exist any designs with wire flexures. Otherwise, we can obtain valid designs with three wire flexures.

For the cases $\hat{C}_x\hat{P}_y$, $\hat{R}_x\hat{R}_y\hat{H}_z$ and $\hat{R}_x\hat{P}_y\hat{H}_z$, we can find at most two reciprocal forces in the constraint space. At last no force wrench exists in the reciprocal constraint space of $\hat{P}_x\hat{P}_y\hat{H}_z$ and $\hat{P}_x\hat{P}_y\hat{P}_z$ as the force component is a zero vector. These cases are shown in Figure 5.

4.5 Four degree-of-freedom motions

Similarly, for $n = 4$ given twists, $m = 2$ linearly independent wrenches are to be found. The results for different motions of this type are given in Table 4. Cases $\hat{C}_x\hat{R}_y\hat{R}_z$ and $\hat{C}_x\hat{R}_y\hat{P}_z$ are realizable. Example designs are shown in Figure 3. Case $\hat{C}_x\hat{H}_y\hat{H}_z$ is realizable if the two pitches p_y, p_z have opposite signs, shown in Figure 4. For cases $\hat{C}_x\hat{R}_y\hat{P}_y$ and $\hat{C}_x\hat{R}_y\hat{H}_z$, only one force can be found in the reciprocal constraint space while there is none for the cases $\hat{C}_x\hat{P}_y\hat{P}_z$ and $\hat{C}_x\hat{P}_y\hat{H}_z$. These four cases are shown in Figure 5.

4.6 Five degree-of-freedom motions

Likewise, for $n = 5$ given twists, there are three cases $\hat{C}_x\hat{C}_y\hat{R}_z$, $\hat{C}_x\hat{C}_y\hat{P}_z$ and $\hat{C}_x\hat{C}_y\hat{H}_z$. See Table 5. Obviously the only realizable case is $\hat{C}_x\hat{C}_y\hat{R}_z$ in which the only removed freedom of

Table 5. Constraint spaces for the five dof motions.

$\prod_T$	The reciprocal force space $\mathcal{F}$	$rank(\mathcal{F})$
$\hat{C}_x\hat{C}_y\hat{R}_z$	$\left(0\ 0\ F_z \mid 0\ 0\ 0\right): F_z \neq 0$	1
$\hat{C}_x\hat{C}_y\hat{P}_z$	$\left(0\ 0\ 0 \mid 0\ 0\ 0\right)$	0
$\hat{C}_x\hat{C}_y\hat{H}_z$	$\left(0\ 0\ 0 \mid 0\ 0\ 0\right)$	0

the functional body is translational and can be realized by a single wire flexure along the translational axis. The design is shown in Figure 3.

5 Conclusion

This paper enumerates all motion spaces that consist of orthogonal motions of the functional stage of a flexure system. Each basic motion can be rotational (R), translation (P) or helical (H) along the coordinate axes of the stage. By enumerating all combinations of these basic motions, we obtain three 1-dof, seven 2-dof, fourteen 3-dof, seven 4-dof, and three 5-dof motion spaces. We find the reciprocal force space $\mathcal{F}$ for each of these 34 motion spaces and determine if they are realizable by checking the rank of $\mathcal{F}$. We conclude that 18 of them are realizable, 4 are realizable only when their pitches have opposite signs and 12 are not realizable. For each of these 34 cases, we provide an example design.

Acknowledgment

The authors knowledge the valuable comments from anonymous reviewers. This work was supported by National Science Foundation under grant CMMI-0845793.

REFERENCES

[1] Smith, S. T., 2000. *Flexure: Element of Elastic Mechanisms*. CRC Press LLC.

[2] Smith, S.T., C. D., 1992. *Foundations of Ultra-Precision Mechanism Design*. Taylor & Francis Books Ltd.

[3] Xu, P., Jingjun, Y., Guanghua, Z., and Shusheng, B., 2008. "The stiffness model of leaf-type isosceles-trapezoidal flexural pivots". *ASME Journal of Mechanical Design,* **130**(8), pp. 082303.1–082303.6.

[4] Howell, L. L., 2001. *Compliant Mechanisms*. Wiley-Interscience, New York, NY.

[5] Blanding, D. L., 1999. *Exact Constraint: Machine Design Using Kinematic Processing*. ASME Press, New York, NY.

[6] Hale, L. C., 1999. "Principles and techniques for designing precision machines". PhD Thesis, MIT, Cambridge, MA.

[7] Awtar, S., and Slocum, A. H., 2007. "Constraint-based design of parallel kinematic xy flexure mechanisms". *ASME Journal of Mechanical Design,* **129**(8), pp. 816–830.

[8] Hopkins, J. B., and Culpepper, M. L., 2010. "Synthesis of multi-degree of freedom, parallel flexure system concepts via freedom and constraint topology (FACT) - part i: Principles". *Precision Engineering,* **34**(2), Apr., pp. 259–270.

[9] Hopkins, J. B., and Culpepper, M. L., 2010. "Synthesis of multi-degree of freedom, parallel flexure system concepts via freedom and constraint topology (FACT). part II: practice". *Precision Engineering,* **34**(2), Apr., pp. 271–278.

[10] Su, H.-J., Dorozhkin, D. V., and Vance, J. M., 2009. "A screw theory approach for the conceptual design of flexible joints for compliant mechanisms". *ASME Journal of mechanisms and robotics,* **1**(4), pp. 041009.1–041009.8.

[11] Hunt, K. H., 1978. *Kinematic Geometry of Mechanisms*. Oxford University Press, New York, NY.

[12] Phillips, J., 1984. *Freedom in Machinery. Volume 1, Introducing Screw Theory*. Cambridge University Press, Cambridge, U.K.

[13] Phillips, J., 1990. *Freedom in Machinery. Volume 2, Screw Theory Exemplified*. Cambridge University Press, Cambridge, U.K.

[14] Lipkin, H., and Duffy, J., 1985. "The elliptic polarity of screws". *Journal of Mechanisms Transmissions and Automation in Design,* **107**(3), pp. 377–386.

[15] Davidson, J. K., and Hunt, K. H., 2004. *Robots and Screw Theory: Applications of Kinematics and Statics to Robotics*. Oxford University Press, New York, NY.

[16] Dai, J. S., and Jones, J. R., 2001. "Interrelationship between screw systems and corresponding reciprocal systems and applications". *Mechanism and Machine Theory,* **36**(5), pp. 633 – 651.

[17] Huang, S., and Schimmels, J. M., 1998. "The bounds and realization of spatial stiffnesses archieved with simple springs connected in parallel". *IEEE Transactions on Robotics and Automation,* **14**(3), pp. 466–475.

[18] Huang, S., 1998. "The analysis and synthesis of spatial compliance". Ph.d. disseration, Marquette University, Milwaukee, WI.

[19] Su, H.-J., and Tari, H., 2010. "On line screw systems and their application to flexure synthesis". In ASME International Design Engineering Technical Conferences. paper no. DETC2010-28361.

[20] Dai, J. S., and Jones, J. R., 2003. "A linear algebraic procedure in obtaining reciprocal screw systems". *Journal of Robotic Systems,* **20**(7).

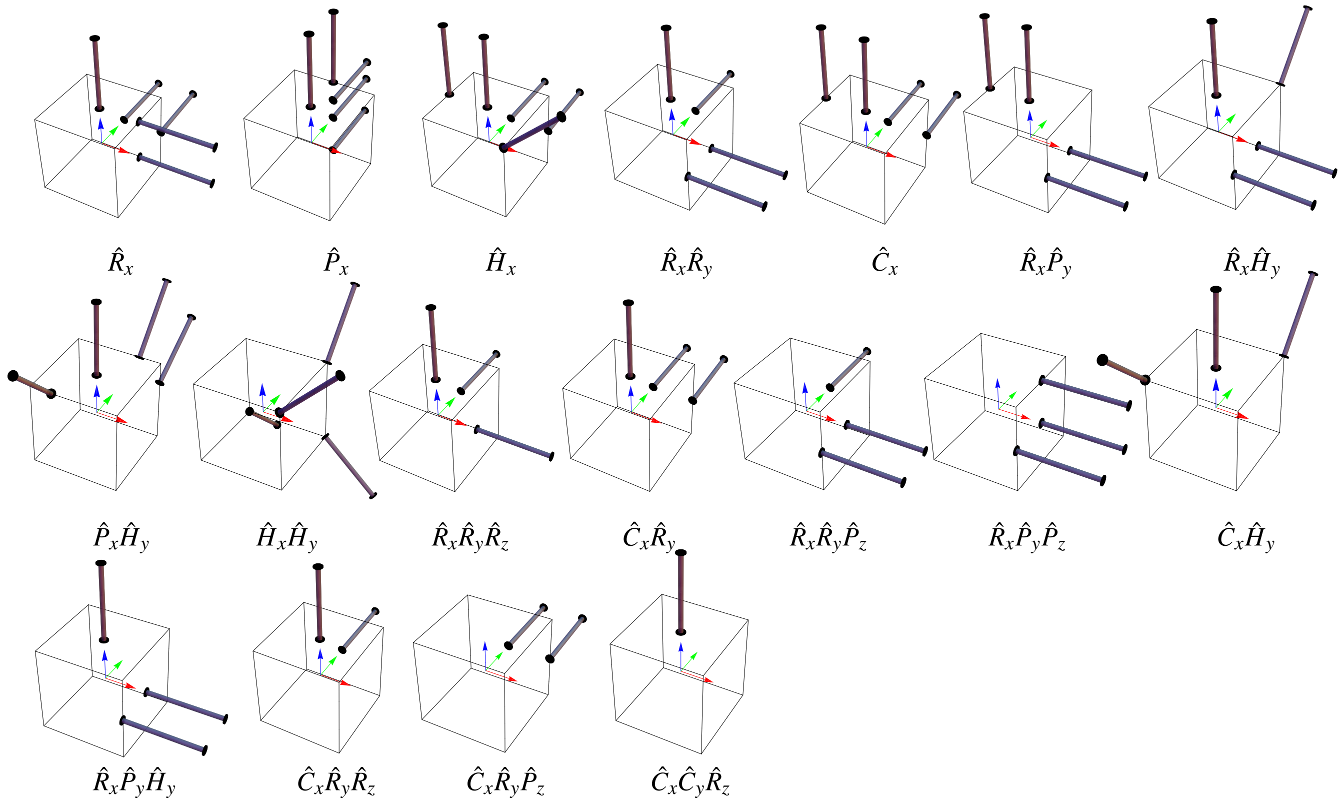

Figure 3. Example designs for the 18 realizable 1-5 dof motion spaces. The box represents the functional body or motion stage. The cylinders represent wire flexures which are welded to the functional body at one end and to the ground at the other.

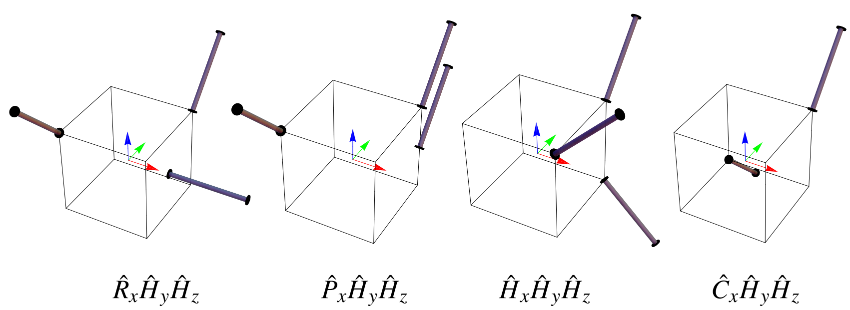

Figure 4. The 4 motion spaces that can be realized only when their pitches have opposite signs.

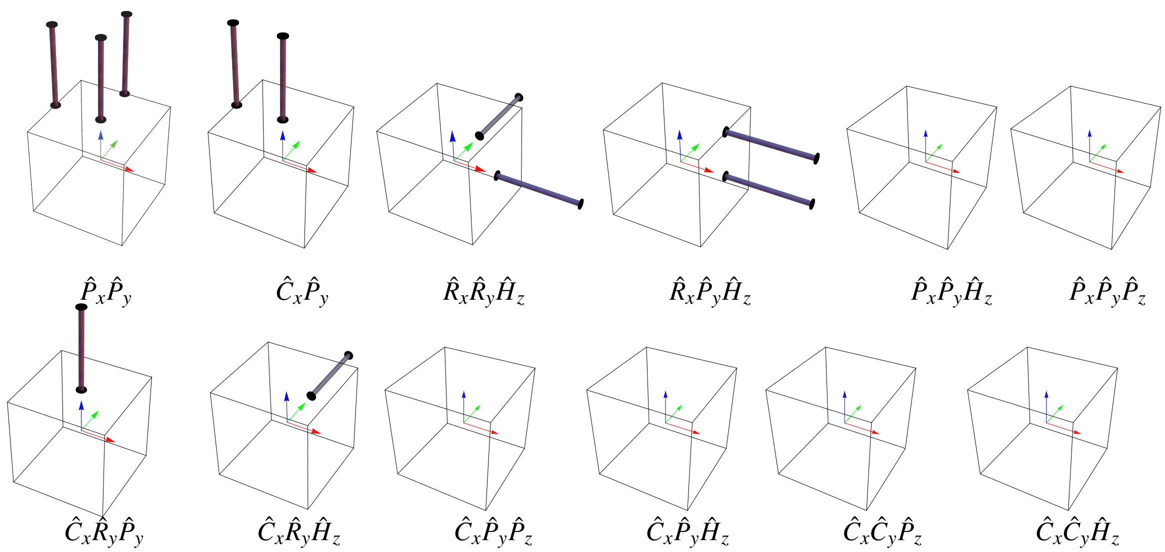

Figure 5. The 12 unrealizable motion spaces.