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SYNTHESIS OF COMPLIANT MECHANISMS WITH SPECIFIED EQUILIBRIUM POSITIONS

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ABSTRACT

This paper presents a synthesis procedure for a compliant four-bar linkage with three specified equilibrium configurations. The finite position synthesis equations are combined with equilibrium constraints at the flexure pivots to form design equations. These equations are simplified by modeling the joint angle variables in the equilibrium equations using sine and cosine functions. Solutions to these design equations were computed using a polynomial homotopy solver. In order to provide a design specification, we first compute the six equilibrium configurations of a known compliant four-bar mechanism. We use these results as design requirements to synthesize a compliant four-bar. The solver obtained eight real solutions which we refined using a Newton-Raphson technique. A numerical example is provided to verify the design methodology.

1 Introduction

Compliant mechanisms are linkage systems designed so elastic deformation of joint and link elements contribute to the effectiveness of the device (Howell 2001). In particular, selectively sizing of link and joint flexures allows a compliant mechanism to be made as an integral structure. This yields advantages in reduced part count, simplified manufacturing, and innovative applications. Hence compliant mechanisms are especially suitable for micro-electro-mechanical-system (MEMS) design.

The design of linkages with elastic elements dates back to Burns and Crossley (1968), who developed a graphical technique to design a four-bar function generator with a flexible coupler link. Sevak and McLarnan (1975) formulated this problem using finite element analysis to model the flexible link. Howell and Midha (1996) introduced the pseudo-rigid body model (PRBM) for compliant four-bar linkages in order to simplify the synthesis of compliant linkages to achieve kinematic specifications. Saggere and Kota (2001) consider the design of a four-bar linkage with flexible driving and driven links that move and deform a flexible coupler between two specified positions and shapes.

A subset of compliant mechanisms that have two or more stable equilibrium configurations are called bistable or multistable compliant mechanisms (MSCM). These systems do not consume power to maintain at equilibrium points and can be used as passive structure in applications such as switches, valves, and clasps. However the existing approaches for design/synthesis of MSCM are complicated and difficult to automate. Jensen et. al (1999) applied PRBM to analyze and design bistable compliant micro-mechanisms. The mechanism parameters were varied to find mechanism configuration with two stable positions. With this approach it is difficult to specify the positions at which the mechanisms is to be stable. King et. al (2004) proposed an optimization based methodology for synthesizing multistable equilibrium systems with specified natural frequencies and stiffness. The complexity of the defining differential equations yield multiple equilibria in the optimization solution.

In this paper, we propose a new systematic methodology

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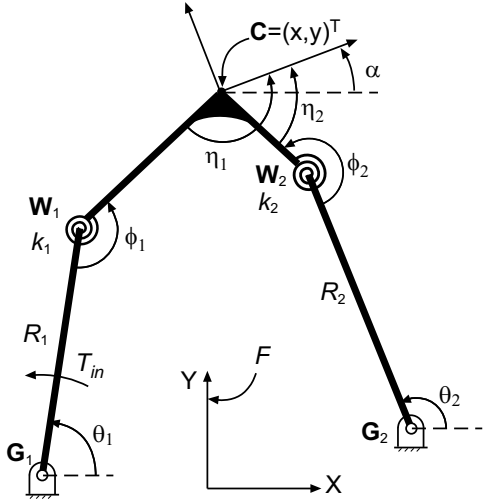


Figure 1. A compliant 4-bar mechanism

for *equilibrium position synthesis problem* of compliant mechanisms. The goal is to determine the dimensions of the mechanism and the spring constant of torsion springs, such that the system has specified equilibrium positions. We first seek for approximately modeling the problem in polynomial equations which we solve using homotopy continuation method. One advantage of homotopy method is that it can obtain all solutions without initial guess. We then perform static analysis to determine the stability of these equilibrium points.

2 Kinematics of the Compliant Four-Bar

Our model of a compliant four-bar linkage shown in Figure 1 is taken from Jensen et. al (1999) and consists of a four-bar linkage with torsion springs connecting the driving and driven cranks to the coupler.

The coordinates of the two ground pivots, $\mathbf{G}_1 = (G_{1x}, G_{1y})^T$ and $\mathbf{G}_2 = (G_{2x}, G_{2y})^T$, are measured relative to the base frame F . The coordinates of the two moving pivots, denoted $\mathbf{w}_1 = (w_{1x}, w_{1y})^T$ and $\mathbf{w}_2 = (w_{2x}, w_{2y})^T$, are measured in the moving coupler frame C . The position of the coupler C relative to F is defined by the rotation angle α and the translation of the origin by the point $\mathbf{C} = (x, y)^T$. The coordinates of the moving pivots \mathbf{W}_i in F are defined by the coordinate transformation

$$\mathbf{W}_i = \begin{Bmatrix} W_{ix} \\ W_{iy} \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{Bmatrix} w_{ix} \\ w_{iy} \end{Bmatrix}, \quad i = 1, 2, \quad (1)$$

which we write as

$$\mathbf{W}_i = \mathbf{C} + [R(\alpha)]\mathbf{w}_i, \quad i = 1, 2, \quad (2)$$

where $[R(\alpha)]$ denotes the 2×2 rotation matrix.

2.1 The position constraint

We model the driving and driven cranks as rigid links of length R_1 and R_2 respectively. If θ_1 and θ_2 are the angles of these cranks, then we have the identities,

$$\mathbf{W}_i = \mathbf{G}_i + R_i \mathbf{e}(\theta_i), \quad i = 1, 2, \quad (3)$$

where $\mathbf{e}(\theta_i) = \begin{Bmatrix} \cos \theta_i \\ \sin \theta_i \end{Bmatrix}$.

Subtract these two equations to define the vector \mathbf{H} along the coupler link from \mathbf{W}_1 to \mathbf{W}_2 , that is

$$\mathbf{H} = [R(\alpha)](\mathbf{w}_2 - \mathbf{w}_1) = \mathbf{G}_2 + R_2 \mathbf{e}(\theta_2) - \mathbf{G}_1 - R_1 \mathbf{e}(\theta_1). \quad (4)$$

This vector is constrained to have a constant magnitude as the linkage moves.

We now compute the first and second derivatives of \mathbf{H} with respect to the drive angle θ_1 . This provides relations that we will need to study the statics of this system.

2.2 The velocity constraint

The first derivative of \mathbf{H} is

$$\frac{d\mathbf{H}}{d\theta_1} = [R'(\alpha)](\mathbf{w}_2 - \mathbf{w}_1)v_1 = R_2 \mathbf{e}^\perp(\theta_2)v_2 - R_1 \mathbf{e}^\perp(\theta_1), \quad (5)$$

where

$$v_1 = \frac{d\alpha}{d\theta_1}, \quad v_2 = \frac{d\theta_2}{d\theta_1}, \quad \text{and } \mathbf{e}^\perp(\theta_i) = \begin{Bmatrix} -\sin \theta_i \\ \cos \theta_i \end{Bmatrix}, \quad i = 1, 2. \quad (6)$$

Notice that $[R'(\alpha)] = [J][R(\alpha)]$, where $[J]$ is the skew-symmetric matrix such that $\mathbf{e}^\perp(\theta_i) = [J]\mathbf{e}(\theta_i)$. This allows us to write equation (5) in the form

$$-[J] \frac{d\mathbf{H}}{d\theta_1} = [R(\alpha)](\mathbf{w}_2 - \mathbf{w}_1)v_1 = R_2 \mathbf{e}(\theta_2)v_2 - R_1 \mathbf{e}(\theta_1), \quad (7)$$

where we have used the identity $-[J][R'(\alpha)] = [R(\alpha)]$. This vector is constrained to have zero magnitude as the linkage moves.

2.3 The acceleration constraint

The second derivative of \mathbf{H} with respect to θ_1 is obtained by computing the derivative of Eq. (7),

$$\begin{aligned} -[J] \frac{d^2\mathbf{H}}{d^2\theta_1} &= [R'(\alpha)](\mathbf{w}_2 - \mathbf{w}_1)v_1^2 + [R(\alpha)](\mathbf{w}_2 - \mathbf{w}_1)\omega_1 \\ &= R_2 \mathbf{e}^\perp(\theta_2)v_2^2 + R_2 \mathbf{e}(\theta_2)\omega_2 - R_1 \mathbf{e}^\perp(\theta_1), \end{aligned} \quad (8)$$

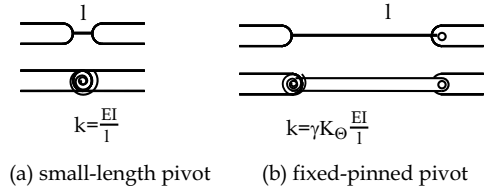


Figure 2. Flexural Joints

where $\omega_1 = \frac{d^2\alpha}{d^2\theta_1}$, $\omega_2 = \frac{d^2\theta_2}{d^2\theta_1}$.

Multiply by $[J]$ to the right of both sides so this equation takes the form

$$\begin{aligned} \frac{d^2\mathbf{H}}{d^2\theta_1} &= -[R(\alpha)](\mathbf{w}_2 - \mathbf{w}_1)v_1^2 + [J][R(\alpha)](\mathbf{w}_2 - \mathbf{w}_1)\omega_1 \\ &= -R_2\mathbf{e}(\theta_2)v_2^2 + R_2\mathbf{e}^\perp(\theta_2)\omega_2 + R_1\mathbf{e}(\theta_1). \end{aligned} \quad (9)$$

This vector must have magnitude zero as the linkage moves.

3 Equilibrium of the Compliant Four-Bar

Howell (2001) shows that the flexures that forms a pivot in a compliant linkage modeled as a hinged joint with an embedded torsion spring. Figure 2 shows two typical flexural joints and their pseudo-rigid body model. For our purposes, we use the “small length pivot” model and define the torsional spring constant $k = \frac{EI}{l}$, where E is modulus of elasticity of the material, I is the area moment of inertia of the cross-section and l is the length of the pivot.

Let the spring constants of our moving pivots, \mathbf{W}_1 and \mathbf{W}_2 be k_1 and k_2 . We also let the rotations at the joints \mathbf{W}_1 and \mathbf{W}_2 be defined by the relative rotation angles ϕ_1 and ϕ_2 , respectively. The vectors $\mathbf{K}_i = \mathbf{C} - \mathbf{W}_i$ are fixed in the moving frame C and make the angle η_i with its x -axis (Figure 1), thus we have

$$\theta_i + \phi_i + \eta_i = \alpha, \quad i = 1, 2. \quad (10)$$

The torque exerted by the torsional springs are given by

$$\tau_i = k_i(\phi_i - \phi_i^0) = k_i\Delta\phi_i, \quad i = 1, 2, \quad (11)$$

where $\Delta\phi_i$ is the angular deflection of the spring.

3.1 Potential energy

The potential energy of torsional springs at the moving pivots (Fig. 1) is given by

$$V = \frac{1}{2}k_1(\Delta\phi_1)^2 + \frac{1}{2}k_2(\Delta\phi_2)^2. \quad (12)$$

The spring deflections are given by

$$\begin{aligned} \Delta\phi_i &= \Delta\alpha - \Delta\theta_i \\ &= (\alpha - \alpha^0) - (\theta_i - \theta_i^0) \quad i = 1, 2, \end{aligned} \quad (13)$$

where α^0 , θ_1^0 , and θ_2^0 determine the undeformed state of the system.

The principle of virtual work allows us to determine that the torque T_{in} applied to the input crank is equal to the derivative of the potential energy, that is

$$\begin{aligned} T_{in} &= \frac{dV}{d\theta_1} = k_1\Delta\phi_1 \frac{d\phi_1}{d\theta_1} + k_2\Delta\phi_2 \frac{d\phi_2}{d\theta_1}, \\ &= k_1\Delta\phi_1(v_1 - 1) + k_2\Delta\phi_2(v_1 - v_2). \end{aligned} \quad (14)$$

This equation is used for *forward static analysis* of a compliant four-bar linkage. Substitute the velocities v_1 and v_2 obtained from (7), to compute the input torque T_{in} required to balance the system.

3.2 Stability

The second derivative of the potential energy (12) defines the stability of an equilibrium point. From (14), we compute

$$\frac{d^2V}{d\theta_1^2} = k_1\Delta\phi_1\omega_1 + k_1(v_1 - 1)^2 + k_2\Delta\phi_2(\omega_1 - \omega_2) + k_2(v_1 - v_2)^2. \quad (15)$$

For our compliant linkage, substitute the velocities (7) and accelerations (9) associated with an equilibrium position into this equation. We have the three cases:

1. If $\frac{d^2V}{d\theta_1^2} > 0$, then the configuration is stable;
2. if $\frac{d^2V}{d\theta_1^2} < 0$, then the configuration is unstable;
3. if $\frac{d^2V}{d\theta_1^2} = 0$, then the configuration is neutrally stable.

The equilibrium points are defined as the configurations such that the mechanism is self-balanced, that is, the input torque T_{in} and the first derivative of the energy (14) equal zero. And if the second derivative of energy is positive, the equilibrium point is a stable one. If a compliant mechanism has two stable equilibrium points, it is called a bistable compliant mechanism (Howell 2001).

4 Inverse Static Analysis

Our goal is to design a compliant four-bar that has a set of specified equilibrium positions. In order to verify the design, we

must be able to determine the equilibrium positions of a given compliant four-bar. This is called *inverse static analysis*. Pigoski and Duffy (1995) and Sun et al. (1997) find the equilibrium positions of a point supported by two and three linear springs, respectively. Su and McCarthy (2004) used polynomial homotopy to find the equilibrium positions of a planar triangle supported by three linear springs. However no torsional springs are involved in above literature. Here we extend our previous work to handle torsional springs for analyzing our compliant four-bar.

Assume the compliant four-bar linkage is known, which means we have values for \mathbf{G}_i , \mathbf{W}_i and R_i , $i = 1, 2$. We can determine the equilibrium positions using the position and velocity constraint equations, (4) and (7), and the equilibrium condition (14), which yields the five equations:

$$\begin{cases} \mathcal{P}_1 \\ \mathcal{P}_2 \end{cases} : [R(\alpha)](\mathbf{w}_2 - \mathbf{w}_1) - \mathbf{G}_2 - R_2 \mathbf{e}(\theta_2) + \mathbf{G}_1 + R_1 \mathbf{e}(\theta_1) = 0, \\ \begin{cases} \mathcal{P}_3 \\ \mathcal{P}_4 \end{cases} : [R(\alpha)](\mathbf{w}_2 - \mathbf{w}_1)v_1 - R_2 \mathbf{e}(\theta_2)v_2 + R_1 \mathbf{e}(\theta_1) = 0, \\ \mathcal{P}_5 : T_{in} - k_1 \Delta\phi_1 (v_1 - 1) - k_2 \Delta\phi_2 (v_1 - v_2) = 0. \end{cases} \quad (16)$$

Rearrange the identities (13) to obtain

$$\theta_i = \alpha - \Delta\phi_i + (\theta_i^0 - \alpha^0), \quad i = 1, 2, \quad (17)$$

and substitute into \mathcal{P}_i , $i = 1, \dots, 5$, to obtain five equations in the five unknowns α , $\Delta\phi_1$, $\Delta\phi_2$, v_1 and v_2 . The translation vector \mathbf{C} to the equilibrium position of C can be determined once these values are known.

The inverse statics equations (16) contain the variables $\Delta\phi_1$ and $\Delta\phi_2$ linearly in the equilibrium equation and as arguments to sine and cosine functions in the kinematics equations. This mixture of terms does not allow us to use a standard polynomial homotopy algorithm to solve these equations. Therefore, we revise the model of the flexure joint as follows.

4.1 Modeling flexure joints

In order to obtain a polynomial form for equations (16), we determine torques $\tau_i = k_i \Delta\phi_i$ as functions of $\sin \Delta\phi_i$ and $\cos \Delta\phi_i$. One way is to approximate the linear function $\tau = k\Delta\phi$ by

$$\tau'(\Delta\phi) = k \sin \frac{\Delta\phi}{2} (c_1 + c_2 \cos \frac{\Delta\phi}{2} + c_3 \cos^2 \frac{\Delta\phi}{2}), \quad (18)$$

where $c_1 = 3.12868$, $c_2 = -1.75512$, $c_3 = 0.65338$ computed by minimizing the error between τ and τ' using the function "Fit" in the software Mathematica. The maximum error is about 1.3% for $\Delta\phi \in [-\pi, \pi]$. The comparison of $\tau(\Delta\phi)$ (solid dots) and $\tau'(\Delta\phi)$ is shown in Figure 3.

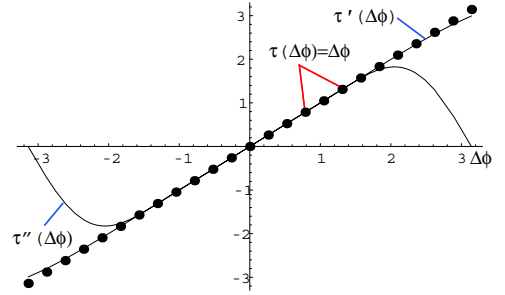


Figure 3. The linear function $\tau = k\Delta\phi$ and the approximating functions τ' and τ'' . The maximum error of τ with τ' and τ'' is 1.3% and 1.5% for $\Delta\phi \in [-\pi/2, \pi/2]$ respectively.

Unfortunately, when this function is substituted into (16) we obtain polynomials of exceptionally high degree. This can be corrected by limiting the deflection angles to the range $\Delta\phi \in [-\pi/2, \pi/2]$, so we can approximate $\tau = k\Delta\phi$ by

$$\tau''(\Delta\phi) = k \sin \Delta\phi (c_1 + c_2 \cos \Delta\phi + c_3 \cos^2 \Delta\phi) \quad (19)$$

where the coefficients $c_1 = 1.56638$, $c_2 = -0.886629$, $c_3 = 0.334932$ are obtained by minimizing the error. The maximum error is about 1.5% for $\Delta\phi \in [-\pi/2, \pi/2]$. The curve $\tau''(\Delta\phi)$ is also shown in Figure 3.

The advantage of using $\tau''(\Delta\phi_i)$ instead of $\tau'(\Delta\phi_i)$ is that the half angle identities are no longer needed, and the resulting polynomial system has a relatively lower degree. Thus, we redefine \mathcal{P}_5 in (16) using τ'' , so

$$\mathcal{P}_5 : T_{in} - \tau''(\Delta\phi_1)(v_1 - 1) - \tau''(\Delta\phi_2)(v_1 - v_2) = 0. \quad (20)$$

Please note that the solutions are valid only if $\Delta\phi_i \in [-\pi/2, \pi/2]$. However this is usually not a limit because most flexural joints do not deflect more than $\pi/2$.

4.2 Solving the inverse statics problem

In order to solve the inverse statics problem we treat $\cos \alpha$, $\sin \alpha$, and $\cos \Delta\phi_i$ and $\sin \Delta\phi_i$ as independent variables denoted c_α , s_α , c_i and s_i respectively. These variables must satisfy the additional polynomial equations,

$$\begin{aligned} \mathcal{P}_6 : c_\alpha^2 + s_\alpha^2 - 1 &= 0, \\ \mathcal{P}_7 : c_1^2 + s_1^2 - 1 &= 0, \\ \mathcal{P}_8 : c_2^2 + s_2^2 - 1 &= 0 \end{aligned} \quad (21)$$

The polynomial system \mathcal{P}_k , $k = 1, \dots, 8$ has a total degree of $2^2 \times 3^2 \times 4^2 \times 2^3 = 4608$. To reduce the number of divergent homotopy path, we examine these polynomials and determine their

monomial structure. Let $\langle x, y \rangle$ denote the linear combination of monomials $ax + by + c$ for arbitrary constants a, b, c , then we have

$$\begin{aligned}
\mathcal{P}_{1,2} &\in \langle c_\alpha, s_\alpha \rangle \langle c_1, s_1, c_2, s_2 \rangle \\
\mathcal{P}_{3,4} &\in \langle c_\alpha, s_\alpha \rangle \langle c_1, s_1, c_2, s_2 \rangle \langle v_1, v_2 \rangle \\
\mathcal{P}_5 &\in \langle s_1, s_2 \rangle \langle c_1, s_1, c_2, s_2 \rangle^2 \langle v_1, v_2 \rangle \\
\mathcal{P}_6 &\in \langle c_\alpha, s_\alpha \rangle^2 \\
\mathcal{P}_7 &\in \langle c_1, s_1 \rangle^2 \\
\mathcal{P}_8 &\in \langle c_2, s_2 \rangle^2
\end{aligned} \tag{22}$$

From this monomial structure, we can compute the generalized linear product (GLP) bound on the number of solutions to be 96, which is much less than the total degree, —for a discussion of GLP bounds and kinematic synthesis see Su et al. (2003). Thus the polynomial homotopy solver tracks 96 paths to find the roots of our polynomial system.

4.3 Polynomial homotopy solvers

Here we summarize briefly the principles underlying polynomial homotopy solvers. Polynomial homotopy algorithms are globally convergent methods for finding all of the isolated solutions to systems of polynomial equations. A homotopy is defined that smoothly transforms the known roots of a specified starting system of polynomials $Q(\mathbf{z}) = 0$, into the desired roots of our system of polynomials $P(\mathbf{z}) = 0$, given by

$$H(\lambda, \mathbf{z}) = (1 - \lambda)Q(\mathbf{z}) + \lambda P(\mathbf{z}), \tag{23}$$

where $\lambda \in [0, 1)$ is the real-valued homotopy parameter. The coefficients of the homotopy are real, but its roots may be complex, therefore we consider $H(\lambda, \mathbf{z})$ to be an array of n complex functions in n complex variables \mathbf{z} together with a single real variable λ .

When $\lambda = 0$, the solutions \mathbf{a}_i to $H(\lambda, \mathbf{z}) = 0$ are the solutions to the start system $Q(\mathbf{z}) = 0$. By varying λ from 0 to 1 and tracing the zero-curve $\gamma(\mathbf{a}_i) = H^{-1}(0)$, we either reach a root of the target system $P(\mathbf{z}) = 0$, or diverge to a root at infinity. Notice that this assumes that the solutions to the start system $Q(\mathbf{z}) = 0$ are known.

The polynomial homotopy algorithm must trace a zero-curve for every root of the start system. Therefore, homotopy solver research has focussed on constructing start systems that provide an improved match to the monomial structure of the target polynomial system. The better the match, the fewer homotopy paths that diverge to infinity.

One convenient way to construct a start system $Q(\mathbf{z})$ is known as the “generalized linear product” or GLP structure (Morgan et. al 1995), which provides an appropriate linear product structure for each polynomial in the target system. The solutions to a GLP start system can be determined by solving sets of

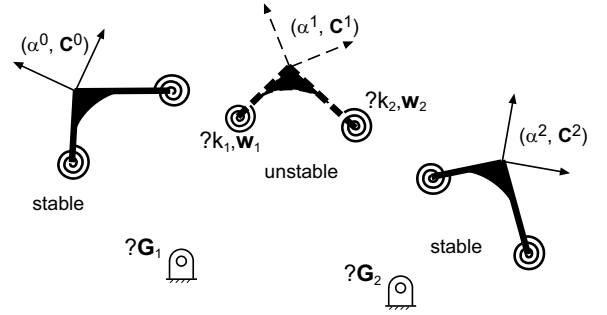


Figure 4. Motion generator of a bistable compliant mechanism. Question marks denote unknown parameters.

linear equations. PHCpack (Verschelde 1997), a publicly available homotopy program, allows the users to specify a GLP start system. If the GLP bound is very large, one could use POLSYS_GLP (Su et. al 2004) on parallel computer systems.

5 Equilibrium Position Synthesis

We now consider the design of a compliant four-bar linkage that reaches three equilibrium positions. In what follows, we use superscript to denote the corresponding parameter at a specific configuration. Let $D^j = (\alpha^j, C^j)$, $j = 0, 1, 2$, be the specified positions of a reference frame C attached to the coupler link. We assume the system is in free state at the initial configuration defined by $D_0 = (\alpha^0, C^0)$. We set the input torque $T_{in} = 0$ because we are interested in the design of bistable mechanisms, that is the system is balanced under no external forces.

Our goal is to find a compliant four-bar such that its coupler is in equilibrium at the specified positions D^j . In other words, we have to determine the fixed pivots \mathbf{G}_i , the moving pivots \mathbf{w}_i in the coupler frame, and the lengths R_i of the drive link 1 and driven link 2, as well as the torsional spring coefficients k_i at the moving pivots. See Figure 4 where question marks denote unknown parameters.

Each vector $\mathbf{W}_i^0 - \mathbf{G}_i$ along crank i in the initial configuration is rotated by the angle $\Delta\theta_i^j = \Delta\alpha^j - \Delta\phi_i^j$ to define $\mathbf{W}_i^j - \mathbf{G}_i$ the position of these cranks in the remaining two configurations. This allows us to define the four vector equations

$$\mathbf{W}_1^j - \mathbf{G}_1 - [R(\Delta\alpha^j - \Delta\phi_1^j)](\mathbf{W}_1^0 - \mathbf{G}_1) = 0, \quad j = 1, 2, \tag{24}$$

$$\mathbf{W}_2^j - \mathbf{G}_2 - [R(\Delta\alpha^j - \Delta\phi_2^j)](\mathbf{W}_2^0 - \mathbf{G}_2) = 0, \quad j = 1, 2. \tag{25}$$

Denote these eight scalar equations as Q_k , $k = 1, \dots, 8$.

Recall that the first position is defined to be in equilibrium, which is equivalent to requiring $\Delta\phi_1^0 = 0$ and $\Delta\phi_2^0 = 0$. The equations that ensure the second and third positions are in equilibrium

are given by (14), that is

$$Q_{8+j}: k_1 \Delta \phi_1^j (v_1^j - 1) + k_2 \Delta \phi_2^j (v_1^j - v_2^j) = 0, j = 1, 2, \quad (26)$$

where v_i^j are the velocities v_i at j th configuration.

Finally, the velocity constraint (7) provides the two vector equations

$$(\mathbf{W}_1^j - \mathbf{W}_2^j) \mathbf{v}_1^j - (\mathbf{W}_1^j - \mathbf{G}_1) + (\mathbf{W}_2^j - \mathbf{G}_2) \mathbf{v}_2^j = 0, \quad j = 1, 2, \quad (27)$$

where we have substitute the identities (2). Denote these four equations as Q_{10+k} , $k = 1, 2, 3, 4$.

The 14 equations formed by (24),(25), (26) and (27) have 18 unknowns consisting of the coordinate vectors \mathbf{G}_1 , \mathbf{G}_2 , \mathbf{w}_1 , and \mathbf{w}_2 , the torsional spring constants k_1 and k_2 , as well as the velocities in the second and third positions $v_1^1, v_1^2, v_2^1, v_2^2$, and the torsion spring deflections $\Delta \phi_1^1, \Delta \phi_1^2, \Delta \phi_2^1$, and $\Delta \phi_2^2$.

5.1 Solve equilibrium synthesis problem

Synthesis of a compliant four-bar with three equilibrium positions requires the solution of 14 equations $Q_k (k = 1, \dots, 14)$ in 18 unknowns. Since we want to apply homotopy solver to obtain the solutions to these equation, we substitute the approximation $\tau''(\Delta \phi_i^j)$ (19) to eliminate the terms $k_i \Delta \phi_i^j$ in Q_9 and Q_{10} .

As was done in the inverse static analysis, we also introduce variables $c_{ij} = \cos \Delta \phi_i^j$ and $s_{ij} = \sin \Delta \phi_i^j$ to represent the sine and cosine functions of the respective angles. The yields four additional polynomial equations

$$c_{ij}^2 + s_{ij}^2 - 1 = 0, \quad i = 1, 2, j = 1, 2. \quad (28)$$

Denote these polynomials as Q_{14+k} , $k = 1, \dots, 4$. The result is 18 polynomial equations in 22 unknowns. Thus, we are free to specify four of the design variables.

For convenience, we specify the spring coefficients k_1 and k_2 and the relative rotation angles $\Delta \theta_1^1$, and $\Delta \theta_1^2$ of the driving link. As a result, the spring deflection $\Delta \phi_1^1, \Delta \phi_1^2$ are calculated by (13). Solving the linear system Q_{1-4} (24) yields the unique solution to vectors $\mathbf{w}_1, \mathbf{G}_1$. This simplification allows us to assemble the remaining design equations.

The GLP start system of the remaining equation can be writ-

ten as

$$\begin{aligned} Q_{5,6} &\in \langle c_{21}, s_{21} \rangle \langle G_{2x}, G_{2y}, w_{2x}, w_{2y} \rangle \\ Q_{7,8} &\in \langle c_{22}, s_{22} \rangle \langle G_{2x}, G_{2y}, w_{2x}, w_{2y} \rangle \\ Q_9 &\in \langle c_{21} \rangle^2 \langle s_{21} \rangle \langle v_1^1, v_2^1 \rangle \\ Q_{10} &\in \langle c_{22} \rangle^2 \langle s_{22} \rangle \langle v_1^2, v_2^2 \rangle \\ Q_{11,12} &\in \langle G_{2x}, G_{2y}, w_{2x}, w_{2y} \rangle \langle v_1^1, v_2^1 \rangle \\ Q_{13,14} &\in \langle G_{2x}, G_{2y}, w_{2x}, w_{2y} \rangle \langle v_1^2, v_2^2 \rangle \\ Q_{17} &\in \langle c_{21}, s_{21} \rangle^2 \\ Q_{18} &\in \langle c_{22}, s_{22} \rangle^2. \end{aligned} \quad (29)$$

The GLP bound is computed as 196 which is much less than the total degree of 16,384. Thus the homotopy solver has to track 196 paths to find solutions of the above 12 equations.

6 Solution Refinement

The solutions to both the inverse static analysis and equilibrium position synthesis problem include a small error due to the approximation (19) used in the equilibrium constraint. We refine these solutions by using them as the initial estimates for a Newton-Raphson root finder for the constraint equations. For the inverse static analysis problem, we use system (16) to refine the homotopy solutions. While for the equilibrium position synthesis problem, we use the system (25), (26) and (27) to refine the solutions. This is done by the Mathematica software function "FindRoot."

7 Examples

Here we study an example compliant mechanism provided by Jensen (1999). We will first perform inverse static analysis to compute all equilibrium points. We pick three of them (two stable and one unstable) to define our design specifications. Then we follow the synthesis procedure presented before to find compliant mechanisms that are in equilibrium at the specified configurations. As a check, the resulting designs must include the original compliant four-bar linkage.

7.1 Inverse static analysis of a compliant four-bar linkage

The system parameters are $\mathbf{G}_1 = (0, 0)^T, \mathbf{G}_2 = (100, 0)^T$, $r_1 = r_2 = 250$, $\theta_1^0 = 83^\circ, \theta_2^0 = 53^\circ$. The length unit is μm . Assume the torsional springs are modeled from small-length flexural joints. The spring coefficients are $k_1 = \frac{E_1 I_1}{l_1} = 29250 \mu N / \mu m$ and $k_2 = \frac{E_2 I_2}{l_2} = 5824.29 \mu N / \mu m$. We also choose $\mathbf{w}_1 = (-112.632, -45.053), \mathbf{w}_2 = (112.632, -45.053)$. The configuration at zero energy is shown in Figure 5(1).

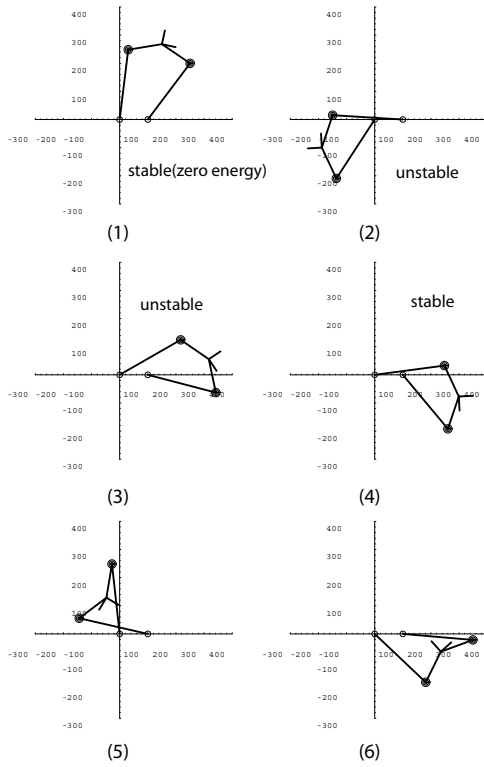


Figure 5. Six equilibrium points of a compliant mechanism. Equilibrium points (1)-(4) belong to one assembly. Equilibrium points (5) and (6) belong to the other assembly

Though the methodology is general for any external torque, we are particularly interested in the case the input torque $T_{in} = 0$ because this will lead us to the design of MSCM.

PHCpack (Verschelde 1997) is used to track all 96 homotopy paths and yield 24 solutions. Six real solutions are shown in Table 1 and Figure 5. Note the first row of each solution is obtained from the homotopy solver and the second row is the refined from the first row using Newton-Raphson's method.

The solutions 1-4 belong to one assembly and solutions 5-6 belong to the other. Please be aware that the configurations on different assembly are not reachable unless the mechanism is reassembled. This phenomenon is usually due to the existence of multi solutions and appears often in mechanism kinematics. See textbook for instance McCarthy (2000) for details. The error of the homotopy solutions 5 and 6 is large because the spring deflections $\Delta\phi_1, \Delta\phi_2$ are off the range $[-90^\circ, 90^\circ]$ which violates our approximation assumption in (19).

By checking the stability condition, we found that solutions 1,4 are stable and solutions 2,3 are unstable. The curves for the energy V and its derivative $\frac{dV}{d\theta_1}$ are shown in Figure 6.

| | θ_1 (°) | θ_2 (°) | α (°) | x | y | $\Delta\phi_1$ (°) | $\Delta\phi_2$ (°) |
|-----|----------------|----------------|--------------|----------|----------|--------------------|--------------------|
| (1) | 83. | 53. | -12.4275 | 150.156 | 267.895 | 0 | 0 |
| | 83. | 53. | -12.4275 | 150.156 | 267.895 | 0 | 0 |
| (2) | 236.865 | 176.434 | 93.273 | -188.064 | -99.4698 | -48.164 | -17.733 |
| | 236.948 | 176.483 | 93.354 | -187.915 | -99.7409 | -48.167 | -17.702 |
| (3) | 29.885 | -14.515 | -56.215 | 316.833 | 56.0063 | 9.3271 | 23.727 |
| | 29.815 | -14.621 | -56.298 | 316.888 | 55.5955 | 9.315 | 23.751 |
| (4) | 7.530 | -50.239 | -86.933 | 298.859 | -77.2987 | 0.964 | 28.734 |
| | 7.556 | -50.195 | -86.892 | 298.923 | -77.1482 | 0.979 | 28.731 |
| (5) | 101.024 | -189.858 | -115.931 | -56.540 | 124.393 | -121.527 | 139.355 |
| | 96.272 | -192.863 | -121.118 | -46.948 | 128.795 | -121.962 | 137.172 |
| (6) | 280.651 | -48.970 | 14.684 | 143.739 | -173.561 | -170.54 | 129.081 |
| | 316.542 | -4.923 | 41.922 | 235.173 | -63.183 | -179.192 | 112.272 |

Table 1. Six equilibrium points of a compliant mechanism. The first row of each solution is obtained from the homotopy solver. The second row is obtain by refining the first row with Newton-Raphson's method.

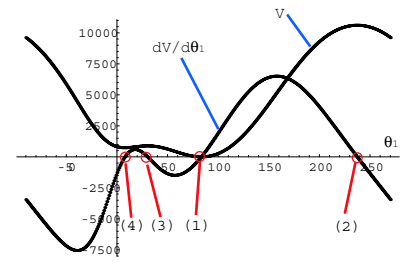


Figure 6. Energy curve with four valid equilibrium points

7.2 Equilibrium synthesis of a compliant four-bar linkage

Now we consider the equilibrium synthesis problem. The design requirements are: (i) the end-effector passes through three equilibrium points (1,3,4) in Table 1 and Figure 5; (ii) the system is stable at equilibrium point 1,4 and unstable at equilibrium point 3.

As mentioned before, we have four free parameters to choose. For instance, we specify the spring constants k_1 and k_2 to be the values in Example 1. In addition, by Table 1, we choose the deflection angles of the input crank $\Delta\theta_1^1 = 29.885^\circ - 83^\circ = -53.185^\circ$ and $\Delta\theta_1^2 = 7.556^\circ - 83^\circ = -75.444^\circ$ to be consistent with Example 1. Using the kinematic constraint (24), we can obtain a unique solution of $\mathbf{G}_1 = (0,0)^T$ and $\mathbf{w}_1 = (112.632, -45.053)^T$.

The remaining equations (29) are solved by PHCpack. It takes 90 seconds to track 196 homotopy path to obtain 64 solutions on a 3Ghz Pentium4 system. The eight real solutions are listed in Table 2. Again the second row of each solution is refined from the first row using Newton-Raphson's method. Figure 7 shows the mechanism and trajectory of the point C by driving the crank θ_1 .

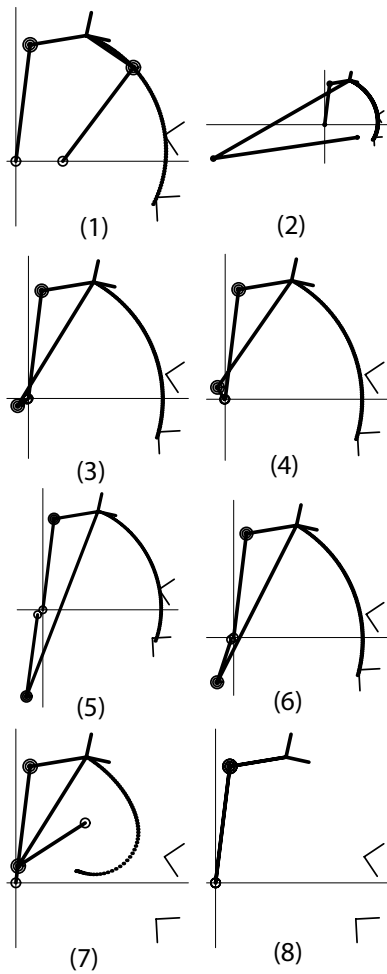


Figure 7. The solutions to the synthesis of a compliant mechanism with three specified equilibrium points

7.3 Evaluation of compliant linkage solutions

As expected, the first solution is the mechanism from Example 1. The solution 2 is a new bi-stable compliant mechanism that satisfies all of the design requirements. See Figure 8 for the energy curves. The three vertical lines represent the specified angles of the input crank at three equilibrium points.

Solutions 3, 5 and 6 satisfy the approximated equilibrium equations but not the exact equations. The error is not reduced to acceptable tolerance. However each of them has a bi-stable feature with equilibriums close the specified task positions.

The solution 4 is stable at θ_1^0 and unstable at θ_1^2 , but not in equilibrium at θ_1^1 . The solution 7 has a branch defect because the linkage does not pass the task positions. In addition, the spring deflections are too large for flexural joints. Lastly solution 8 is a degenerate open chain mechanism.

| Sol. | G_{2x} | G_{2y} | w_{2x} | w_{2y} | $\Delta\phi_2^1(^{\circ})$ | $\Delta\phi_2^2(^{\circ})$ | Check |
|------|----------|----------|----------|----------|----------------------------|----------------------------|---------------|
| (1) | 100.158 | -0.090 | 113.936 | -43.915 | -23.623 | -28.594 | example 1 |
| | 100. | 0 | 112.632 | -45.053 | -23.751 | -28.731 | |
| (2) | 197.84 | -77.204 | -707.932 | -641.932 | 11.128 | 16.778 | new design |
| | 198.097 | -77.266 | -704.726 | -640.257 | 11.183 | 16.863 | |
| (3) | -2.584 | -2.408 | -109.975 | -298.322 | 174.457 | -0.516 | approx. |
| | -2.834 | -2.481 | -109.573 | -315.824 | 92.007 | -0.924 | |
| (4) | -3.924 | -0.697 | -111.964 | -296.23 | -176.648 | -8.316 | approx. |
| | -2.097 | -0.413 | -111.731 | -271.481 | -78.329 | -4.714 | |
| (5) | -13.847 | -12.823 | -83.1707 | -530.29 | 7.015 | -4.805 | approx. |
| | -14.269 | -13.3209 | -81.7627 | -534.448 | 6.779 | -4.926 | |
| (6) | -6.269 | -5.019 | -104.312 | -409.838 | 18.250 | -2.859 | approx. |
| | -6.163 | -4.920 | -104.564 | -406.821 | 18.796 | -2.843 | |
| (7) | 60.623 | 50.7616 | -90.676 | -231.392 | 161.821 | 179.476 | branch defect |
| | 147.8 | 127.631 | -91.635 | -258. | 65.813 | 89.973 | |
| (8) | 0 | 0 | -112.632 | -45.053 | -9.315 | -0.979 | degenerate |
| | 0 | 0 | -112.632 | -45.0529 | -9.315 | -0.979 | |

Table 2. Eight compliant mechanisms for reaching three specified equilibrium points. The first row of each solution is obtained from the homotopy solver. The second row is obtain by refining the first row with Newton-Raphson's method.

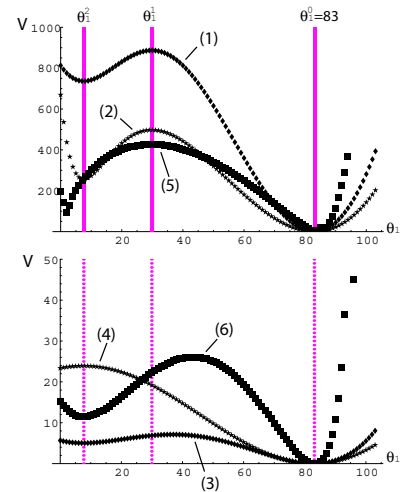


Figure 8. Energy curves of the solutions 1-6 for equilibrium point synthesis

8 Conclusion

This paper presents a new formulation of equilibrium position synthesis of compliant mechanisms. To demonstrate the methodology, we focus on a compliant four-bar linkage with two grounded rigid joints and two floating flexural joints. The linear torque displacement functions at the joints of this chain are approximated by a harmonic formula to allow solution of the design and equilibrium equations using a polynomial homotopy solver. A Newton-Raphson refinement routine is used to remove the effects of this approximation. The result is the ability to design compliant mechanisms with specified equilibrium configurations. The methodology was verified by ensuring that a known design was among the solutions combined with a complete in-

verse static analysis of this design.

One importance of this result lies in that both kinematic and static constraints of compliant mechanisms can be modeled in polynomial equations. As a result, methodologies, theories and algorithms, such as homotopy continuation, resultant elimination that were developed in traditional rigid body kinematics, can be applied to the research of compliant mechanisms.

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