Synthesis of Bistable Compliant Four-bar Mechanisms using Polynomial Homotopy

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Abstract

In this paper we formulate and solve the synthesis equations for a compliant four-bar linkage with three specified equilibrium configurations in the plane. The kinematic synthesis equations as for rigid body mechanisms are combined with equilibrium constraints at the flexure pivots to form design equations. These equations are simplified by modeling the joint angle variables in the equilibrium equations using sine and cosine functions. Polynomial homotopy continuation is applied to compute all of the design candidates that satisfy these design equations, which are refined using a Newton-Raphson technique. A numerical example demonstrates design methodology in which the homotopy solver obtained eight real solutions. Two of them provide two stable and one unstable equilibrium, hence can be used as the prototype of bistable compliant mechanisms.

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1. INTRODUCTION

Compliant mechanisms are linkage systems designed so that elastic deformation of joint and link elements contribute to the effectiveness of the device [1]. In particular, selectively sizing of link and joint flexures allows a compliant mechanism to be made as a single integral structure. This provides the advantages of reduced part count, simplified manufacturing, and innovative applications. Hence, compliant mechanisms are especially suitable for micro-electro-mechanical-system (MEMS) design.

The design of linkages with elastic elements dates back to Burns and Crossley [2], and later Sevak and McLarnan [3]. Recently, Saggere and Kota [4] and Fecker et al. [5] developed synthesis technique for linkage systems with elastic links.

An important problem in the design of compliant linkages is the synthesis of linkages that have two or more stable equilibrium configurations. These systems maintain their configuration at stable equilibrium points without consuming power, and can be used as passive structures in applications such as switches, valves, and clasps. Jensen et al. [6–8] use the pseudo-rigid body model (PRBM) approach to analyze and design bistable compliant micro-mechanisms. They varied the mechanism parameters to find mechanisms with two stable positions. King et al. [9] used the fully elastic model and optimization to define multi-stable equilibrium systems with specified natural frequencies and stiffness. Unfortunately, these methods do not allow the specification of the linkage configurations at which the stable equilibria occurs.

Our synthesis problem begins with specifying a set of equilibrium positions for a compliant mechanism. And the goal is to determine dimensions and the spring constants of flexural joints such that the mechanism achieves static equilibrium at specified positions in the absence of external forces. This problem evolutes from “rigid body guidance” problem which was introduced by Burmester [10] who used a graphical approach to seek a planar four bar to reach five specified planar positions. A modern formulation [12] of this problem yields five quadratic equations in five unknown parameters, which can be solved by various methods such as polynomial homotopy or resultant elimination.

Homotopy method, also called “continuation method,” for solving polynomial systems were first proposed by Garcia and Zangwill [13]. The idea is based on the fact that small changes on the coefficients of the polynomial systems will result small changes to the roots.

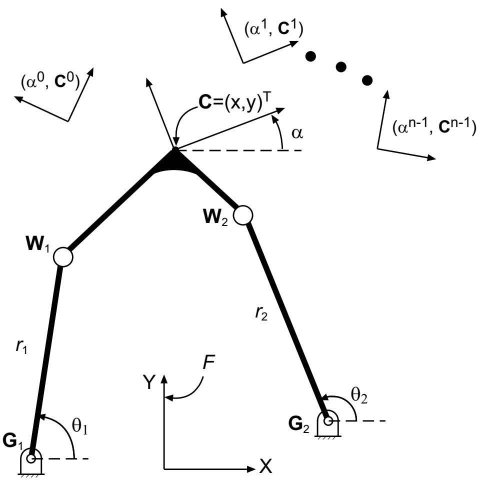

FIG. 1: Synthesis of a rigid body four-bar to reach n specified task positions

Starting at roots of a "start system,", homotopy method traces the solution along the the so-called "homotopy path," as the start system is continuously transformed into the target systems. Compared with iteration based methods, homotopy method can find all solutions and is globally convergent without an initial guess to the solutions.

In what follows we first revisit the synthesis of a rigid body four-bar to reach a set of specified task positions. We then formulate the design of a compliant four-bar with specified equilibrium positions using the pseudo-rigid body model of Howell and Midha [14]. Our design equations are formed by combining geometric constraints and equilibrium constraints. A new aspect of our formulation is the transformation of the equilibrium equations into polynomial form through an approximation of flexural joints so that we can apply polynomial solution techniques to solve the design equations. This method is especially beneficial when a compliant mechanism undergoes large deflections.

2. KINEMATIC SYNTHESIS OF A FOUR-BAR LINKAGE

A rigid four-bar mechanism (Fig. 1) is formed by connecting two RR (revolute-revolute) dyads at the base and the couple link. The position of the coupler C relative to the fixed frame F is defined by the rotation angle α and the translation of the origin by the point $\mathbf{C} = (x, y)^T$. The length and angle of the two RR dyads are denoted by r_1, r_2 and θ_1, θ_2 respectively. The coordinates of the two grounded pivots, $\mathbf{G}_1 = (G_{1x}, G_{1y})^T$ and $\mathbf{G}_2 = (G_{2x}, G_{2y})^T$, are measured relative to the base frame F. The coordinates of the two moving pivots, denoted $\mathbf{w}_1 = (w_{1x}, w_{1y})^T$ and $\mathbf{w}_2 = (w_{2x}, w_{2y})^T$, are measured in the moving coupler frame C.

2.1. Geometric constraint equations

The coordinates of the moving pivots $\mathbf{W}_i$ in F are defined by the coordinate transformation

$$\mathbf{W}_i = \mathbf{C} + [R(\alpha)]\mathbf{w}_i, \quad i = 1, 2, \tag{1}$$

where $[R(\alpha)]$ denotes the 2×2 rotation matrix of angle α. The kinematic constraint equations of a four-bar can be written as

$$\mathbf{W}_i - \mathbf{G}_i - r_i\mathbf{e}(\theta_i) = 0, \quad i = 1, 2, \tag{2}$$

where $\mathbf{e}(\theta_i) = \begin{Bmatrix} \cos\theta_i \\ \sin\theta_i \end{Bmatrix}$. The result is a set of four equations with five unknowns $\theta_1, \theta_2, x, y, \alpha$. By specifying one of the angles θ_1, θ_2 as the driver, we can obtain the other four unknowns by solving system (2).

To obtain velocity constraint, we compute the first derivative of (2) with respect to θ_1 and subtract one from the other and yield

$$(\mathbf{W}_1 - \mathbf{W}_2)\nu_1 - (\mathbf{W}_1 - \mathbf{G}_1) + (\mathbf{W}_2 - \mathbf{G}_2)\nu_2 = 0, \tag{3}$$

where $\nu_1 = \frac{d\alpha}{d\theta_1}, \nu_2 = \frac{d\theta_2}{d\theta_1}$ are angular velocities.

2.2. Geometric synthesis equations

Given a set of n task positions $D^j = (\alpha^j, \mathbf{C}^j)(j = 0, 1, \ldots, n-1)$ for the coupler link, the goal of synthesis is to find the dimensions of the linkage. The synthesis equations can be obtained by writing the kinematic constraint equations (2) for each of the task positions, that is

$$\mathbf{W}_1^j - \mathbf{G}_1 - r_1\mathbf{e}(\theta_1^j) = 0, \quad j = 0, 1, \ldots, n-1, \tag{4}$$

$$\mathbf{W}_2^j - \mathbf{G}_2 - r_2\mathbf{e}(\theta_2^j) = 0, \quad j = 0, 1, \ldots, n-1, \tag{5}$$

where $\mathbf{W}_i^j = \mathbf{C}^j + [R(\alpha^j)]\mathbf{w}_i$ and θ_i^j represent the position of the moving joints and the angle of RR dyads at each of the task positions.

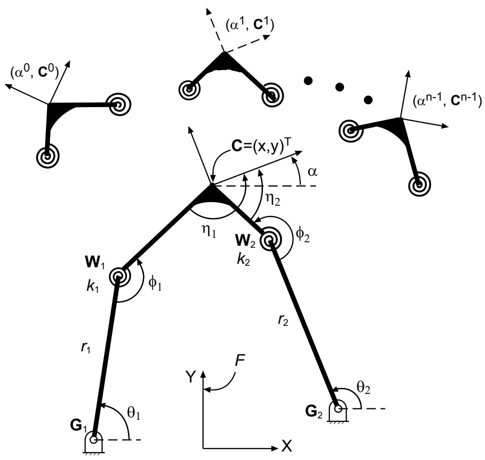

FIG. 2: Synthesis of a compliant four-bar to reach n specified equilibrium positions

Eliminating r_i and θ_i^0 from (4,5) yields

$$\mathbf{W}_1^j - \mathbf{G}_1 - [R(\Delta\theta_1^j)](\mathbf{W}_1^0 - \mathbf{G}_1) = 0, \quad j = 1, \ldots, n-1, \tag{6}$$

$$\mathbf{W}_2^j - \mathbf{G}_2 - [R(\Delta\theta_2^j)](\mathbf{W}_2^0 - \mathbf{G}_2) = 0, \quad j = 1, \ldots, n-1, \tag{7}$$

where $\Delta\theta_i^j = \theta_i^j - \theta_i^0$ represent the relative rotation angles of the task positions.

Eq(6,7) is a system of $4(n-1)$ scalar equations in $2(n-1)+8$ unknowns: relative angles $\Delta\theta_i^j$, and four pivots $\mathbf{w}_1, \mathbf{w}_2, \mathbf{G}_1, \mathbf{G}_2$. The number of specified task positions must satisfy $4(n-1) \leq 2(n-1)+8$ or $n \leq 5$. If less than five positions are specified, the designers will have freedom to choose some of the design parameters. For this case, numerous techniques have been developed to help designers picking right values for the free choices. The typical strategy is applying additional constraints, e.g. Grashof condition, to the design problem. For example, in the software SphinxPC [11], all possible values for the single free choice are calculated and graphically represented in a type map which users can interactively pick values for the free parameter. As a result, users can rapidly explore the solution space and find ideal values for the free choices.

Note it is possible to further eliminate relative angles $\Delta\theta_i^j$ and achieve a system with only four pivots as unknowns [12]. However we choose to keep them because they will show up in static equilibrium equations for the compliant mechanism synthesis.

3. EQUILIBRIUM SYNTHESIS OF COMPLIANT FLEXURES

Figure 2 shows a compliant four-bar linkage which differs the rigid four-bar in Fig. 1 in that the two moving pivots are embedded with a torsional spring. This linkage can be the pseudo-rigid-body-model [1] of a compliant linkage. We use k_1 and k_2 to represent the spring constants of the flexural joints. And the deflection of the torsional springs are determined by relative rotation angles ϕ_1 and ϕ_2.

3.1. Equilibrium constraint equations

The potential energy of the compliant four-bar is given by

$$V = \frac{1}{2}k_1(\Delta\phi_1)^2 + \frac{1}{2}k_2(\Delta\phi_2)^2. \tag{8}$$

The spring deflections are given by

$$\begin{aligned}\Delta\phi_i =& \Delta\alpha - \Delta\theta_i \\ =& (\alpha - \alpha^0) - (\theta_i - \theta_i^0) \quad i = 1, 2,\end{aligned} \tag{9}$$

where α^0, θ_1^0, and θ_2^0 determine the undeformed state of the system.

The principle of virtual work tells us that the linkage is balanced if and only if the derivative of the potential energy with respect to a general coordinate equals zero, that is

$$\frac{dV}{d\theta_1} = k_1\Delta\phi_1\frac{d\phi_1}{d\theta_1} + k_2\Delta\phi_2\frac{d\phi_2}{d\theta_1} = 0,$$

which we can rewrite as

$$k_1\Delta\phi_1(\nu_1 - 1) + k_2\Delta\phi_2(\nu_1 - \nu_2) = 0. \tag{10}$$

The equilibrium constraint equation (10) together with kinematic constraint equation (2) and velocity constraint equations (3) form a system of five equations in five unknowns $\theta_1, \theta_2, \nu_1, \nu_2, \alpha$. The problem of inverse static analysis seeks for equilibrium positions that satisfy these equations. In our previous work [15], we applied polynomial homotopy method to solve these equations to find all equilibrium positions. To determine the stability of the equilibrium position, one can check the second derivative of the potential energy. And if the second derivative is positive, the equilibrium position is stable. If a compliant mechanism has two or more stable equilibrium positions, it is called a bistable or multi-stable compliant mechanism.

3.2. Equilibrium synthesis equations

Now we consider the synthesis of the compliant four-bar linkage to reach a set of specified equilibrium positions. As in the synthesis of a rigid body four-bar, we also use $D^j = (\alpha^j, \mathbf{C}^j)(j = 0, 1, \ldots, n-1)$ to denote the specified equilibrium positions of a reference frame C attached to the coupler link. We assume the system is in free state at the initial configuration defined by $D_0 = (\alpha^0, \mathbf{C}^0)$. Our goal is to find compliant four-bar linkages that can statically balance at the specified positions D^j. More specifically, we need determine the fixed pivots $\mathbf{G}_i$, the moving pivots $\mathbf{w}_i$ in the coupler frame, the lengths r_i of the driver link 1 and driven link 2, and the torsional spring constants k_i.

a. Geometric constraints As in the synthesis of rigid body mechanisms (6,7), we write the kinematic synthesis equations for each of the specified task positions D^j. This allows us to define the $2(n-1)$ vector or $4(n-1)$ scalar equations

$$\mathcal{K}_{1j}: \quad \mathbf{W}_1^j - \mathbf{G}_1 - [R(\Delta\alpha^j - \Delta\phi_1^j)](\mathbf{W}_1^0 - \mathbf{G}_1) = 0, \quad j = 1, \ldots n-1, \tag{11}$$

$$\mathcal{K}_{2j}: \quad \mathbf{W}_2^j - \mathbf{G}_2 - [R(\Delta\alpha^j - \Delta\phi_2^j)](\mathbf{W}_2^0 - \mathbf{G}_2) = 0, \quad j = 1, \ldots n-1, \tag{12}$$

where we have substituted the identities $\Delta\theta_i^j = \Delta\alpha^j - \Delta\phi_i^j$ from (9).

b. Equilibrium constraints Since the first position is assumed to be the free state (natural configuration), the equilibrium condition is automatically satisfied due to zero spring deflections, $\Delta\phi_1^0 = \Delta\phi_2^0 = 0$. The equations that ensure the other positions are in equilibrium can be obtained by writing (10) for each position, that is

$$\mathcal{Q}_j: \quad k_1\Delta\phi_1^j(\nu_1^j - 1) + k_2\Delta\phi_2^j(\nu_1^j - \nu_2^j) = 0, \quad j = 1, \ldots n-1, \tag{13}$$

where ν_i^j are the angular velocities ν_i at jth task position.

c. Velocity constraints Finally, we write the velocity constraint (3) at each position to provide another $n-1$ vector or $2(n-1)$ scalar equations

$$\mathcal{V}_j: \qquad (\mathbf{W}_1^j - \mathbf{W}_2^j)\nu_1^j - (\mathbf{W}_1^j - \mathbf{G}_1) + (\mathbf{W}_2^j - \mathbf{G}_2)\nu_2^j = 0, \quad j = 1, \ldots, n-1. \tag{14}$$

Eq (11-14) is a system of $7(n-1)$ scalar equations with $9+4(n-1)$ independent unknowns: the coordinate vectors $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{w}_1$, and $\mathbf{w}_2$, the torsional spring constants k_1 and k_2, the velocities ν_1^j, ν_2^j, and the spring deflections $\Delta\phi_1^j, \Delta\phi_2^j$. Please note we only count k_1, k_2 as one unknown because only their ratio matters in (13). The number of specified equilibrium positions must satisfy $7(n-1) \leq 9+4(n-1)$ or $n \leq 4$. If less than four equilibrium positions are specified, the designers will have freedom to choose some of the design parameters.

As in the synthesis of rigid body mechanisms, additional techniques are yet to be developed to help designers choosing right values for the free choices. It seems possible to apply techniques developed in rigid body mechanism synthesis in compliant mechanism synthesis. An interactive simulation and design system will be critical for the design of compliant mechanisms. Virtual reality are one of the key enabling techniques. Nevertheless this lies in the scope of future work.

4. POLYNOMIAL HOMOTOPY SOLUTION

The equilibrium constraint equations (13) contain the variables $\Delta\phi_i^j$ linearly and kinematic constraint equations (11,12) contain their sine and cosine functions. This mixture of terms does not allow us to use a standard polynomial homotopy algorithm to solve these equations. Therefore, we transform the equilibrium equations (13) as follows.

We approximate the torque $k_i\Delta\phi_i^j$ in (13) using the following formula

$$k_i\Delta\phi_i^j \approx k_i \sin\Delta\phi_i^j(c_1 + c_2\cos\Delta\phi_i^j + c_3\cos^2\Delta\phi_i^j) \quad i = 1,2; j = 1,\dots,n-1, \tag{15}$$

where the coefficients are obtained by minimizing the error. The maximum error is only 1.5% for $\Delta\phi_i^j \in [-\pi/2, \pi/2]$ which is sufficient for our problem. To accommodate even large deflections, one may consider half angle formulation. See our previous work [15] for details.

Finally, we introduce variables $c_{ij} = \cos\Delta\phi_i^j$ and $s_{ij} = \sin\Delta\phi_i^j$ to represent the sine and cosine functions of the respective angles. The yields additional $2(n-1)$ polynomial equations

$$\mathcal{P}_{ij}: \quad c_{ij}^2 + s_{ij}^2 - 1 = 0, \quad i = 1,2; j = 1,\dots,n-1. \tag{16}$$

If $n = 4$ equilibrium positions are specified, equations (11-14,16) form a system of $9(n-1) = 27$ polynomial equations with 27 unknowns: $\mathbf{G}_i$, $\mathbf{w}_i$, ν_i^j, c_{ij}, s_{ij} and ratio of spring constants k_1/k_2.

If $n = 3$ equilibrium positions are specified, we will have $9(n-1) = 18$ equations in 21 unknowns. Thus, we are free to specify three of the design variables. For instance, we can specify the ratio of the spring constants k_1/k_2 and the relative rotation angels $\Delta\theta_1^1$, and $\Delta\theta_1^2$ of the driving link. As a result, the spring deflection $\Delta\phi_1^1, \Delta\phi_1^2$ are calculated by (9). Solving the linear system (11) yields a unique solution to vectors $\mathbf{w}_1$, $\mathbf{G}_1$. This simplification allows us to assemble the remaining design equations.

In what follows, we will apply homotopy method for solving the remaining 12 design equations. Polynomial homotopy is a globally convergent method for finding all the isolated solutions to systems of polynomial equations. However before applying the homotopy method to our equations, we must first construct an appropriate start system. One convenient way to construct a start system is known as the *generalized linear product* or GLP structure, which provides an appropriate linear product structure for each polynomial in the target system. The solutions to a GLP start system can be determined by solving sets of linear equations, —for a discussion of GLP bounds and kinematic synthesis see [16, 17].

By observing the monomial structure of the remaining 12 equations, we construct the GLP start system as following

$$\begin{aligned}
\mathcal{K}_{21} &\in \langle c_{21}, s_{21}\rangle\langle G_{2x}, G_{2y}, w_{2x}, w_{2y}\rangle \\
\mathcal{K}_{22} &\in \langle c_{22}, s_{22}\rangle\langle G_{2x}, G_{2y}, w_{2x}, w_{2y}\rangle \\
\mathcal{Q}_{1} &\in \langle c_{21}\rangle^2\langle s_{21}\rangle\langle \nu_1^1, \nu_2^1\rangle \\
\mathcal{Q}_{2} &\in \langle c_{22}\rangle^2\langle s_{22}\rangle\langle \nu_1^2, \nu_2^2\rangle \\
\mathcal{V}_{1} &\in \langle G_{2x}, G_{2y}, w_{2x}, w_{2y}\rangle\langle \nu_1^1, \nu_2^1\rangle \\
\mathcal{V}_{2} &\in \langle G_{2x}, G_{2y}, w_{2x}, w_{2y}\rangle\langle \nu_1^2, \nu_2^2\rangle \\
\mathcal{P}_{21} &\in \langle c_{21}, s_{21}\rangle^2 \\
\mathcal{P}_{22} &\in \langle c_{22}, s_{22}\rangle^2,
\end{aligned} \tag{17}$$

where $\langle , \rangle$ represents the monomial formed by linear combination of the variables in the list. The GLP bound of this start system is computed as 196 which is much less than the total degree of $16,384$. Thus the homotopy solver has to track only 196 paths to find solutions of the above 12 equations. Notice that equations $\mathcal{K}_{21}, \mathcal{K}_{22}, \mathcal{P}_{21}, \mathcal{P}_{22}$ are vector equations. Hence they should be counted as two equations each.

Due to the approximation (15), the solutions returned from the homotopy solver will include a small error. This can be fixed by applying Newton-Raphson root finder with the homotopy solutions as the initial estimates. This is done by the Mathematica software function "FindRoot."

position	orientation	stability
$\mathbf{C}^0 = (150.156, 267.895)$	$\alpha^0 = -12.4275°$	stable(free state)
$\mathbf{C}^1 = (-187.915, -99.7409)$	$\alpha^1 = 93.354°$	unstable
$\mathbf{C}^2 = (298.923, -77.1482)$	$\alpha^2 = -86.892°$	stable

TABLE I: Three specified equilibrium points.

5. SYNTHESIS FOR THREE EQUILIBRIUM POSITIONS

Here we use a numerical example provided in [6, 15] to verify our approach. The design requirements are: (i) the coupler link passes through three equilibrium points as shown in Table I; (ii) ideally the system is stable at equilibrium point 0, 2 and unstable at equilibrium point 1. And we assume the first design position is the free state, that is the compliant joints are not deflected.

As mentioned before, we have three free parameters to choose for this particular problem. Here we specify the spring constants $k_1 = 29250\mu N/\mu m$ and $k_2 = 5824.29\mu N/\mu m$ as in [15]. Again here we actually specify only one parameter because only their ratio matters. The other two free choices are the deflection angles of the input crank $\Delta\theta_1^1 = -53.185°$ and $\Delta\theta_1^2 = -75.444°$. Using the kinematic constraint (11), we can obtain an unique solution of $\mathbf{G}_1 = (0, 0)^T$ and $\mathbf{w}_1 = (112.632, -45.053)^T$.

The remaining equations (17) are solved by PHCpack [18]. It takes 90 seconds to track 196 homotopy path to obtain 64 solutions on a 3Ghz Pentium4 system. The eight real solutions are listed in Table II. The second row of each solution is refined from the first row using Newton-Raphson's method. Figure 3 shows the mechanism and trajectory of the point $\mathbf{C}$ by driving the crank θ_1.

As expected, the first solution is the mechanism provided in reference [6, 15]. The solution 2 is a new bi-stable compliant mechanism that satisfies all of the design requirements. See Figure 4 for the energy curves. The three vertical lines represent the specified angles of the input crank at three equilibrium points. The local minimum and maximum of the energy curve represent stable and unstable equilibrium respectively.

Solutions 3,5 and 6 satisfy the approximated equilibrium equations but not the exact equations. The error is not reduced to acceptable tolerance. However each of them has a bi-stable feature with equilibrium close to the specified task positions.

Sol.	G_{2x}	G_{2y}	w_{2x}	w_{2y}	$\Delta\phi_2^1(°)$	$\Delta\phi_2^2(°)$	Check
(1)	100.158	-0.090	113.936	-43.915	-23.623	-28.594	
	100.	0	112.632	-45.053	-23.751	-28.731	known sol.
(2)	197.84	-77.204	-707.932	-641.932	11.128	16.778	
	198.097	-77.266	-704.726	-640.257	11.183	16.863	new design
(3)	-2.584	-2.408	-109.975	-298.322	174.457	-0.516	
	-2.834	-2.481	-109.573	-315.824	92.007	-0.924	approx.
(4)	-3.924	-0.697	-111.964	-296.23	-176.648	-8.316	
	-2.097	-0.413	-111.731	-271.481	-78.329	-4.714	approx.
(5)	-13.847	-12.823	-83.1707	-530.29	7.015	-4.805	
	-14.269	-13.3209	-81.7627	-534.448	6.779	-4.926	approx.
(6)	-6.269	-5.019	-104.312	-409.838	18.250	-2.859	
	-6.163	-4.920	-104.564	-406.821	18.796	-2.843	approx.
(7)	60.623	50.7616	-90.676	-231.392	161.821	179.476	branch
	147.8	127.631	-91.635	-258.	65.813	89.973	defect
(8)	0	0	-112.632	-45.053	-9.315	-0.979	
	0	0	-112.632	-45.0529	-9.315	-0.979	degenerate

TABLE II: Eight compliant mechanisms for reaching three specified equilibrium points. The first row of each solution is obtained from the homotopy solver. The second row is obtain by refining the first row with Newton-Raphson's method.

The solution 4 is stable at θ_1^0 and unstable at θ_1^2, but not in equilibrium at θ_1^1. The solution 7 has a branch defect because the linkage passes one task position only. In addition, the spring deflections of flexural joints are too large and violates flexural approximation. Lastly solution 8 is a degenerated open chain mechanism.

6. CONCLUSION AND FUTURE WORK

This paper presents a new formulation of equilibrium position synthesis of a compliant four-bar linkage with two grounded rigid body joints and two floating flexural joints. The linear torque displacement functions at the joints of this chain are approximated by a har-

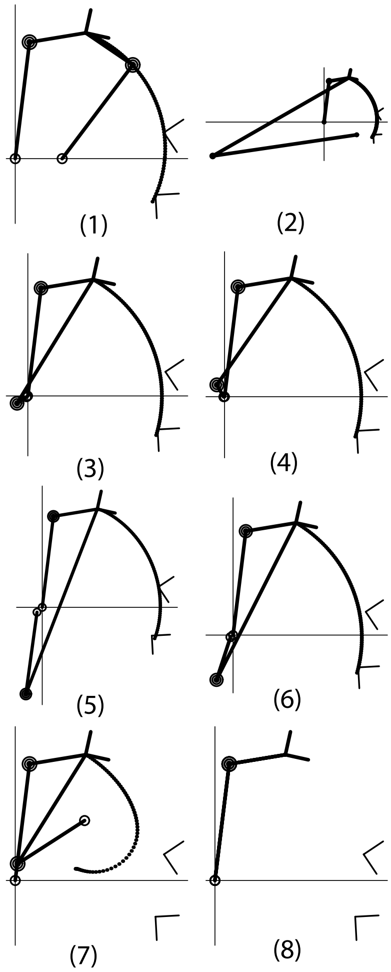

FIG. 3: The solutions to the synthesis of a compliant mechanism with three specified equilibrium points

monic formula to allow solution of the design and equilibrium equations using a polynomial homotopy solver. A Newton-Raphson refinement routine is used to remove the effects of this approximation. The result is the ability to design compliant mechanisms with specified equilibrium configurations. The methodology was verified by ensuring that a known design was among the solutions combined with a complete inverse static analysis of this design.

If less than four task positions are specified, designers have to choose values for the free choices. An interactive simulation and design system is extremely important for facilitating designers picking right values for the free choices. Currently at Iowa State University, a virtual environment for compliant mechanism design is under development. This system will allow designers interactively explore solution space through virtual reality technology including immersive display and haptic devices.

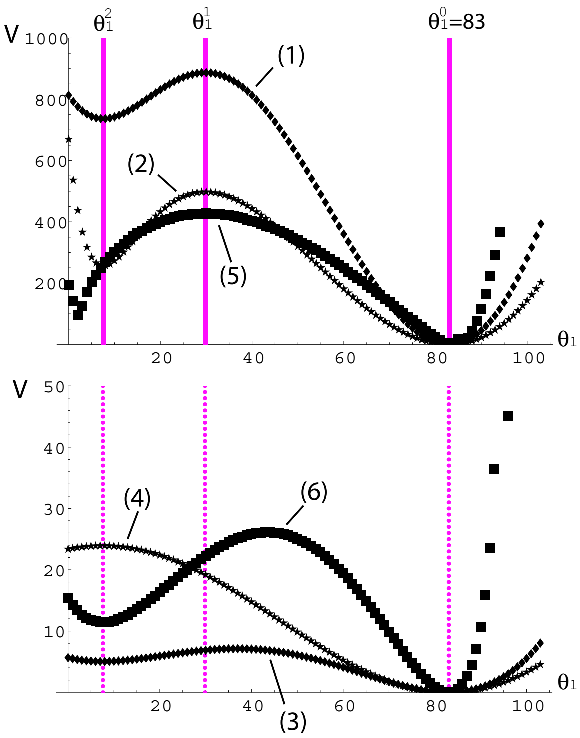

FIG. 4: Energy curves of the solutions 1-6 for equilibrium point synthesis

7. ACKNOWLEDGMENT

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