

A Constrained Homotopy Technique for Excluding Unwanted Solutions From Polynomial Equations Arising in Kinematics Problems

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Abstract

This paper presents a constrained technique which enables classical homotopy continuation solvers to exclude unwanted solutions from a system of polynomials. It is well known that extraneous and degenerate solutions often appear in polynomial equations derived from kinematics problems. These solutions usually do not have physical meanings and prevent us from understanding of the intrinsic feature of the studied problem. Traditionally to exclude these solutions, we have to rely on a tedious post-processing technique which often suffers from numerical instability especially when the number of unwanted solutions is large. In the proposed constrained homotopy technique, the unwanted solutions are mapped to the solutions at infinity of a higher dimensional system. We show that the extra polynomials do not increase the homotopy path count when they are used with a classical homotopy solver. This convenient and user friendly technique does not require

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any changes to the homotopy codes or extra steps to post-process out unwanted solutions. Compared with the regeneration procedure, our approach offers more freedom in eliminating positive dimensional solutions. In addition, our numerical experiments show that the solutions obtained with the constrained technique seem to have a better accuracy than the post processing approach. Kinematics analysis and synthesis problems are employed to demonstrate that the technique is capable of eliminating unwanted isolated and positive dimensional solutions. In the revisit of the classical nine-point path generation of four-bar, we verify that the maximum number of distinct mechanisms is 1442 triplets.

Key words: Solving Polynomial Systems, Homotopy Continuation Method, Kinematic Analysis, Kinematic Synthesis, Extraneous Solutions, Degenerate Solutions.

1. Introduction

The kinematics problems of mechanisms normally result in systems of polynomial equations. Raghavan and Roth [1] recommended three major methods: dialytic elimination, Gröbner Bases and polynomial continuation for obtaining all solutions to these polynomial systems. In this paper, we focus on the polynomial continuation due to its capability of systematically solving relatively large systems.

The polynomial homotopy or continuation method, first proposed by [2], starts at roots of a “start system” and traces roots along the so-called “homotopy paths” as the start system is continuously transformed into the target system. With the recent development of mathematical theories [3, 4, 5, 6], especially the advent of probability-one homotopy, the polynomial homotopy method has become a reliable tool for finding all isolated solutions

to a polynomial system. Recent work has been shifted towards lowering the number of paths that must be tracked to find all the finite solutions (solutions at infinity being of no practical interest). In essence, all the methods try to construct a start system for the homotopy map that better models the structure of the given polynomial system, the target system for the homotopy map. The most often used start systems are the linear product based start systems including 1-homogeneous, multi-homogeneous, partitioned linear product (PLP), generalized linear product (GLP). In the middle of 1990's, a major computational breakthrough emerged in solving deficient polynomial systems. The new method, called the *polyhedral homotopy* continuation method [7], takes a great advantage of the combinatorial root count, called *mixed volume*, in the Bernshtein's theorem [8], which generally provides a much tighter bound for the number of isolated zeros of a polynomial system. Li [5] provided a comprehensive survey on the polyhedral homotopy. We call these homotopy methods "classical homotopy methods" as their focus is on finding isolated solutions of a square (number of equations equals the number of variables) polynomial system.

Recently extensive works have been done to adapt the homotopy method to identify positive dimensional solutions of a square or a non-square polynomial system. One approach is the regeneration technique [9] which solves the polynomial system equation by equation. Slicing the given system of polynomials each time with a general linear space of complementary dimension leads to witness solutions on positive dimensional algebraic sets. In this paper, we call this method a "non-classical homotopy" method. The regeneration technique is very useful in sifting out the degenerate (positive dimensional) solutions of many kinematic synthesis problems.

After years of development, many homotopy packages are freely avail-

able. The linear product based homotopy codes include HomPack [10], POLSYS_PLP [11], POLSYS_GLP [12] and so on. HOM4PS2 [13, 14] and PHoM [15] are the two latest polyhedral homotopy solvers. Bertini [16] supports the regeneration technique for finding isolated and positive dimensional solutions and is capable of dealing with non-square systems of polynomials [17, 18, 19]. PHCpack [20] is the most versatile code supporting various linear product homotopy, polyhedral homotopy and the equation by equation procedure. Lee, Li and Tsai [13] reported that HOM4PS2 is generally the fastest package for solving small to moderately large sparse systems.

In the last decades, the homotopy continuation method has been proved to be a powerful and effective tool to solve kinematics problems. Roth and Freudenstein [21] were the first applying “bootstrap” technique which was a kind of immature numerical continuation method to solve kinematics problems. Tsai and Morgan [22] applied a homotopy solver to solve the inverse kinematics of a general 6R manipulator. Their result stimulated the derivation of the resultant elimination solutions of [23, 24] of this classical problem. Another milestone problem is the forward kinematics of the Stewart-Gough platform. Using the homotopy continuation method, Raghavan [25] was the first to numerically report that there are at most 40 generic configurations of a general Stewart-Gough platform. Later Wampler [26] reported the same root count by using soma coordinators or dual quaternions. These results on root counting guided the elimination solutions by several authors [27, 28]. A seminar review on the use of homotopy continuation methods in solving kinematics problems can be found in [29, 30].

It is well known that extraneous and degenerate solutions often appear in the polynomial equations obtained in solving kinematic analysis and synthesis problems. These solutions are usually not the solutions engineers

are looking for. The existence of these unwanted solutions hinders the understanding of the intrinsic feature of studied kinematics problems, such as maximum number of assemblies in kinematic analysis problems. A well known example was the solution of the kinematics of 7R spatial loop by [31]. They reduced the problem to a single polynomial of degree 32. However 16 of them are extraneous solutions. Lee and Liang [23] reduced the problem to a polynomial of 16th degree in late 1980s. Another example is the nine-point path generation of four-bar problem. Wampler et al. [32] provided a complete homotopy solution to this problem by using a 2-homogeneous start system. After a careful post-processing step that checks each solution against a specified threshold and retracts suspicious paths, they reported that only 4326 out of 143,360 solutions are nondegenerate.

In this paper, we present a new constrained homotopy technique for excluding a set of unwanted (isolated or positive dimensional) solutions from a polynomial system. The idea is to augment the original system by extra constraint polynomials. Essentially the unwanted solutions (which can be the union or intersection of a set of algebraic sets) are mapped to solutions at infinity in the augmented system. Some recent advances in homotopy theory on sifting out higher dimensional solutions are given in [18], but the new technique is not currently implemented in any of the homotopy codes mentioned above. Our technique is very convenient and allows the user to easily exclude unwanted solutions by simply appending more equations to the system. It does not require any change of homotopy codes or additional programming to post-process out unwanted solutions. One important feature of this technique is that the number of homotopy paths to be tracked remains unchanged when it is used together with a classical homotopy solver. Our experimental runs show that computation time using this technique is

moderately the same as when the technique is not used.

The paper is organized as the following. In section 2, the constraint and augmented polynomials are defined. Section 3 introduces the use of the constraint polynomials with various homotopy methods including multi-homogeneous, general linear product and polyhedral homotopy. In Section 4, we use two practical kinematics problems to demonstrate the effectiveness of the constrained homotopy in excluding unwanted extraneous or degenerate solutions. Finally, summary and conclusions are given.

2. The Constraint Polynomials and the Augmented System

Given a system of n polynomials in n unknowns $x = (x_1, \dots, x_n) \in \mathbb{C}^n$,

$$P(x) = \begin{cases} p_1(x_1, \dots, x_n) \\ \vdots \\ p_n(x_1, \dots, x_n) \end{cases}, \quad (1)$$

we define the *affine variety* of $P(x)$ or the solution set of equations $P(x) = 0$ by

$$V(P) = \{x \in \mathbb{C}^n \mid p_i(x) = 0, i = 1, \dots, n\}. \quad (2)$$

The affine variety $V(P)$ is a set of algebraic entities which can be points (zero dimensional), lines (one dimensional), surfaces (two dimensional) and so on. Typically for engineering problems, we are interested in finding only zero dimensional or isolated solutions of $P(x) = 0$. However very often $V(P)$ may consist of unwanted solutions such as extraneous (typically isolated) solutions due to an inappropriate derivation process or degenerate (typically positive dimensional) solutions which are meaningless to engineers.

2.1. *The constraint polynomial for the union of algebraic sets*

Let us define the algebraic set of the “unwanted solutions” by $V(G)$ with $G(x) \in \mathbb{C}^n$ written as,

$$G(x) = \prod_{j=1}^m g_j(x) \quad (3)$$

where g_j represent the algebraic sets to be excluded from $V(P)$. Obviously $V(G) = \cup_{j=1}^m V(g_j)$.

To exclude the solutions of $G(x) = 0$ in (3) from $V(P)$, we augment the original system (1) by introducing a new complex variable $l \in \mathbb{C}$ and a new polynomial $p_{n+1}(x, l)$ in \mathbb{C}^{n+1} , written as

$$\mathcal{G}(x, l) : p_{n+1}(x, l) = 1 - lG(x) = 1 - l \prod_{j=1}^m g_j(x). \quad (4)$$

We call $p_{n+1}(x, l)$ a “constraint polynomial”.

2.2. *The constraint polynomial for the intersection of algebraic sets*

Suppose we want to exclude the algebraic sets intersected by polynomials $g_j(x) (j = 1, \dots, m)$, denoted by $V(g_1, \dots, g_m) = \cap_{j=1}^m V(g_j)$. That is we are interested in finding the finite isolated point set S given by

$$S = \{x_1, \dots, x_s\} = V(P) \setminus V(g_1, \dots, g_m) \quad (5)$$

where s is finite and “ \setminus ” is the set minus operator defined below,

$$A \setminus B = \{x \in \mathbb{C}^n | x \in A, x \notin B\}. \quad (6)$$

For this case, we define $G(x)$ by

$$G(x) = \sum_{j=1}^m c_j g_j(x), \quad (7)$$

where $c_j \neq 0$ are randomly chosen complex coefficients. Clearly with probability one, we have $G(x) \neq 0, \forall x \in S$. And $G(x) = 0, \forall x \in V(g_1, \dots, g_m)$.

Similarly for the intersection of algebraic sets, the constraint polynomial can be written as

$$\mathcal{G}(x, l) : p_{n+1}(x, l) = 1 - lG(x) = 1 - l \sum_{j=1}^m c_j g_j(x). \quad (8)$$

If all the algebraic sets vanish, i.e. $g_j(x) = 0 (j = 1, \dots, m)$, the variable l will be at infinity. On the other hand, if at least one of $g_j(x) \neq 0$, we have $\sum_{j=1}^m c_j g_j(x) \neq 0$ in general since c_j are randomly chosen. Hence we can almost always obtain a finite solution of l with probability one.

2.3. The augmented polynomial system

For both the union and the intersection of algebraic sets, we define the “augmented polynomial” system $\hat{P}(x, l)$ as

$$\hat{P}(x, l) = \begin{cases} P(x) \\ \mathcal{G}(x, l) \end{cases} \quad (9)$$

which is solved with homotopy codes to obtain the desired solution set

$$V(P) \setminus V(G) = \{x \in \mathbb{C}^n \mid P(x) = 0, G(x) \neq 0\}. \quad (10)$$

Theorem 1. *The augmented polynomial system has the same solution set as the original system $V(P)$ excluding the solution set of $G(x)$, i.e. $\pi(V(\hat{P})) = V(P) \setminus V(G)$, where $\pi : (x, l) \mapsto (x)$ is the operator that projects (x, l) to (x) by neglecting the solution of l .*

PROOF. For any set of finite values of $(x', l') \in V(\hat{P})$, we have $\hat{P}(x', l') = 0$, i.e. $P(x') = 0$ and $\mathcal{G}(x', l') = 0$ by definition. The former implies $x' \in V(P)$ and the latter implies $G(x') \neq 0$ i.e. $x' \notin V(G)$ for both union and

intersection cases. Therefore $x' \in V(P) \setminus V(G)$. This shows $\pi(V(\hat{P})) \subset V(P) \setminus V(G)$.

On the other hand, for any set of finite values of $x' \in V(P) \setminus V(G)$, we have $P(x') = 0$ and $G(x') \neq 0$. We can solve Eq. (4) or (8) to obtain the solution of $l' = 1/G(x')$. Obviously we have $(x', l') \in V(\hat{P})$, hence $V(P) \setminus V(G) \subset \pi(V(\hat{P}))$ neglecting l' . Combining with the first part proves the theorem. \square

2.4. Additional considerations

Suppose we intend to exclude both the union of a set of m_1 polynomials $g_j(x) = 0$ and the intersection of another set of m_2 polynomials $h_k(x) = 0$, we would introduce two new variables $l = (l_1, l_2)$ and write the constraint polynomials as

$$\mathcal{G}(x, l) = \begin{cases} p_{n+1}(x, l) = 1 - l_1 \prod_{j=1}^{m_1} g_j(x) \\ p_{n+2}(x, l) = 1 - l_2 \sum_{k=1}^{m_2} c_k h_k(x) \end{cases} \quad (11)$$

Although the constraint polynomial $\mathcal{G}(x, l)$ (4) excludes the unwanted solutions given by $G(x) = 0$ in theory, it has a serious numerical issue in practice. Due to the limit of the digits used in computer programs, it is difficult to determine if $\mathcal{G}(x, l)$ is close to zero when the number of unwanted solution components, m , is large. Based on the working precision of arithmetic, we use a small real number ϵ as the tolerance for determining if each g_j goes to zero, i.e. checking $\|g_j\| < \epsilon$. We would have to use a very tight tolerance ϵ^m to detect whether $\mathcal{G}(x, l)$ goes to zero or not, i.e. checking $\|\prod_{j=1}^m g_j\| < \epsilon^m$. When m is large, we would have to adopt an arithmetic with a much higher precision such as the adaptive precision. However this will significantly increase the computational burden.

To overcome numerical difficulties mentioned above, it would be better to introduce m new variables $l = (l_1, \dots, l_m) \in \mathbb{C}^m$ and augment the original polynomial system by m dimensions. Mathematically, the newly added polynomials are written as

$$\mathcal{G}(x, l) = \begin{cases} p_{n+1}(x, l) = 1 - l_1 g_1(x) \\ \vdots \\ p_{n+m}(x, l) = 1 - l_m g_m(x) \end{cases} \quad (12)$$

It is not hard to prove that Theorem 1 still holds for the new constraint equations (12). Please note that these extra equations increase computational burden to the homotopy solver. However a carefully designed tracker that takes advantage of the special structures of $\mathcal{G}(x, l)$ can effectively minimize the effect on computation performance. More importantly, system (12) has a better numerical robustness compared with system (4). From now on, this paper will only discuss the augmented system (12) which is used to solve our kinematics examples.

2.5. Example

Let us consider the following system of two polynomial equations,

$$P(x, y) : \begin{cases} p_1 = (x - y^2)(x - 1) \\ p_2 = (x - y^2)(y - 2) \end{cases} \quad (13)$$

Figure 1(a) shows the plots of the polynomials p_1 and p_2 in the 2D space. It is obvious to see that $V(P)$ is comprised of a point $\{x = 1, y = 2\}$ and a parabola $\{x = y^2\}$. Suppose we want to exclude the solution set $\{x = y^2\}$ from the system. We define the following constraint polynomial:

$$p_3(x, y, l_1) = 1 - l_1(x - y^2) = 0 \quad (14)$$

Figure 1(b) shows the plots of the augmented polynomial system in the three dimensional space. Although in the two dimensional space, p_1 and p_2 intersect at the parabolic curve $g_1(x, y) = x - y^2$, they do not intersect with the constraint polynomial p_3 at this curve in the 3D space as the parameter l_1 goes to infinity. This leaves the only finite solution, a point

$$\{x = 1, y = 2, l_1 = -1/3\}$$

which is in $V(P)$ neglecting the solution of l_1 .

Suppose we would like to further exclude the intersection of $h_1(x, y) = (x - 1)$ and $h_2(x, y) = (y - 2)$. We can augment the system for the following constraint polynomial

$$p_4(x, y, l_2) = 1 - l_2[c_1(x - 1) + c_2(y - 2)] = 0 \quad (15)$$

where c_1 and c_2 are randomly chosen complex coefficients. Of course, there will be no solution left for this case.

3. The Constrained Homotopy

In this section, we give an overview on the homotopy method and describe how the constrained technique is used with the existing homotopy solvers. The classical homotopy continuation method for solving $P(x) = 0$ is to find a trivial system $Q(x) = (q_1(x), \dots, q_n(x))$ and then follow the solution curves in the real variable λ from $\lambda = 0$ to $\lambda = 1$ which make up the solution set of

$$H(x, \lambda) = (1 - \lambda)\gamma Q(x) + \lambda P(x) = 0 \quad \text{with generic } \gamma \in \mathbb{C} \setminus \{0\}.$$

More precisely, all the isolated solutions of $P(x) = 0$ can be found if the system $Q(x) = 0$, known as the start system satisfies the three properties listed in [5].

It is well known that polynomial systems arising in kinematics applications are usually sparse. One key step in speeding up the solution process with the homotopy method is to find an optimal start system with a minimum root count. Sommese and Wampler [6] listed a hierarchy of structures for constructing a start system. Here we discuss three kinds of homotopies: multi-homogeneous homotopy, general linear product homotopy and polyhedral homotopy. For each homotopy, we discuss the start systems of the constraint polynomials. Also we show that the root count of the augmented system equals to that of the original system which is a very important key for computational efficiency.

3.1. Start systems for the constraint polynomials

In Multi-homogeneous or simply m -homogeneous start system, we partition the unknowns x into k mutually exclusive sets:

$$S = \{S_1, \dots, S_k\}, \quad \text{with} \quad \bigcup_{j=1}^k S_j = \{x_1, \dots, x_n\} \quad \text{and} \quad S_i \cap S_j = \emptyset, i \neq j \quad (16)$$

When the constraint polynomials $\mathcal{G}(x, l)$ are used to augment the original system $P(x)$, we only need to augment the partition set S by the following set

$$S_{k+1} = \{l_1, \dots, l_m\} \quad (17)$$

In the general linear product (GLP) homotopy, we define a set structure S_i for each polynomial $p_i(x)$

$$S_i = \{S_{i1}, S_{i2}, \dots, S_{ik_i}\}, \quad \text{with} \quad S_{ij} \subset \{x_1, \dots, x_n\}, \quad i = 1, \dots, n \quad (18)$$

where k_i is the number of subsets of S_i and S_{ij} are not mutually exclusive in general. And the set structure for the constraint polynomial $\mathcal{G}(x, l)$ in

(12) is obtained by augmenting the set structure of $g_j(x)$ with a new subset $\{l_j\}$, written as

$$S_{n+j} = \mathcal{S}_j \cup \{l_j\}, \quad j = 1, \dots, m \quad (19)$$

where \mathcal{S}_j is the set structure of $g_j(x)$. Please note that \mathcal{S}_j can be chosen as its 1-homogenous structure as it is not relevant to the constrained homotopy. See the root count of the augmented GLP start system in the following section.

The *polyhedral homotopy* continuation method takes a great advantage of the combinatorial root count, called *mixed volume* which generally provides a much tighter bound for the number of isolated zeros of a polynomial system. When polyhedral homotopy is employed, the number of homotopy paths that need to be traced agrees with the mixed volume of the polynomial system. Therefore, when the mixed volume of a polynomial system is far less than its total degree, the method will greatly reduce the extraneous paths and thereby considerably limit the wasteful computations.

For both m -homogeneous and GLP homotopy, the start system can be automatically constructed once the set structure is provided. This is already implemented in existing homotopy codes such as Bertini, POLSYS_GLP and PHCpack. And the polyhedral homotopy codes such as HOM4PS can construct the start system based on the monomial structure of the augmented system. Hence no extra data is needed for polyhedral homotopy codes.

3.2. Root count of the augmented start system

As mentioned earlier, the root count of the start systems also known as the Bézout number which is an upper bound of the root count of the polynomial system to be solved. It is also the number of homotopy paths to be tracked. The lower this root count, the faster the solution process.

Our numerical experiments show that the root count of the augmented start system for m -Homogeneous, general linear product and polyhedral homotopy equals that of the original system. This is due to the fact that each constraint polynomial consists of only one linear term of one variable l_i . For the sake of conciseness, here we only provide a short proof for the GLP start system.

Theorem 2. *The GLP start system constructed from the augmented set structure has the same root count as the start system of the original set structure.*

PROOF. By definition [6], the Bézout number of a GLP start system is calculated by counting the admissible combinations of linear factors. Here admissible means that the linear system formed by taking a linear factor from each polynomial of the start system must be nonsingular. Considering the set structure of the constraint polynomial given in (19), we can see that the new variables $l = (l_1, \dots, l_m)$ only appear linearly but not in any of (18). To obtain admissible combinations, we must pick the set $\{l_i\}$ from S_{n+i} . This leaves us the admissible combinations of S_i which count to the root count of the original start system. \square

This theorem holds true for m -homogeneous start system as it can be considered as a special case of the GLP start system. And a complete proof for polyhedral homotopy is provided in Appendix. These theorems are very important because they imply that the homotopy solver using an m -Homogeneous or a GLP start system or polyhedral homotopy for solving the augmented polynomial system will track the same number of paths as in solving the original polynomial system.

3.3. Example

Let us consider the two equations in Eq.(13) and calculate the root count for each start system. The partition for the 3-homogeneous start system of the augmented system is

$$S_1 = \{x\}, S_2 = \{y\}, S_3 = \{l_1\}$$

We can obtain the 3-homogeneous Bézout number of 8 which equals the 2-homogeneous Bézout number of the original system. And when the augmented 3-homogeneous start system is used, tracking these 8 homotopy paths leads us to the only finite solution $\{x = 1, y = 2\}$ which we seek.

In the GLP start system, we choose the following set structures for the augmented system

$$S_1 = \{\{x\}, \{y\}, \{x, y\}\}, S_2 = \{\{y\}^2, \{x, y\}\}, S_3 = \{\{y\}, \{x, y\}, \{l_1\}\}$$

where the power is the set degree. And the GLP Bézout number of the augmented start system is counted to 7. We applied the homotopy code POLSYS_GLP to solve the $P(x, y)$ without using the constraint polynomial. The result shows that only one path leads to the isolated solution $\{x = 1, y = 2\}$. The other six paths land on the parabolic curve $\{x = y^2\}$. See Figure 2. When the constraint polynomial is used, POLSYS_GLP [12] obtained a single finite solution $\{x = 1, y = 2, l_1 = -1/3\}$ after tracing 7 solution paths. To show the details of the path tracking process, we plot the Euclidean norm of the variables vs. the homotopy parameter λ in Figure 3. Clearly six out of seven paths are directed to solutions at infinity.

Mixed volume calculation for the two dimensional system (13) can be calculated by hand using the Minkowski sum of the convex hulls. Figure 4 (a) and (b) represent the mixed volumes of the polynomials $p_1(x, y)$ and

$p_2(x, y)$ respectively. The mixed volume of the system (13) in C^n is the total volume, in this case area, of the Minkowski sum but subtracting the original mixed volumes which is $11 - 3 - 2 = 6$. The shaded area in Figure 4 (c) gives the mixed volume of the polynomial system (13). This hand calculation is consistent with the result obtained by the polyhedral homotopy solver HOM4PS2. When the system is augmented by the extra polynomial $p_3(x, y, l_1)$ in (15), it is not convenient to visualize the calculation. Instead we utilized the polyhedral homotopy solver HOM4PS2 which reported a mixed volume of 6.

The results above show that the augmented start system has the same root count as that of the original start system for m -Homogeneous, GLP and polyhedral homotopy..

4. Kinematics Examples

In what follows, we will use two kinematics problems to demonstrate the use of the constrained homotopy technique in solving polynomial systems. The first example is the inverse kinematics of an RR dyad in which we show the elimination of extraneous (isolated) solutions. And the second example is the well known nine-point path generation of the planar four-bar in which we demonstrate the elimination of degenerate (positive dimensional) solutions.

4.1. Inverse kinematics of an RR dyad

Let's consider the inverse kinematics problem of an RR dyad consisting of two pin joints as shown in Figure 5. The problem is to find θ_j for which for the predefined constant lengths of the links the tip point P reaches the coordinates (x, y) . It is easy to verify that the kinematics equations of the

robot arm take the following form:

$$\begin{bmatrix} a_1 \cos(\theta_1) + a_2 \cos(\theta_2) - x \\ a_1 \sin(\theta_1) + a_2 \sin(\theta_2) - y \end{bmatrix} = 0, \quad (20)$$

where constants a_j are the link lengths of the dyad, defined in the Figure 5.

One common practice for converting (20) into a polynomial system is to substitute the following trigonometric transformations,

$$\sin(\theta_j) = \frac{2t_j}{1+t_j^2} \quad \text{and} \quad \cos(\theta_j) = \frac{1-t_j^2}{1+t_j^2}$$

and clearing denominators to obtain

$$\begin{bmatrix} p_1 = a_1(1-t_1^2)(1+t_2^2) + a_2(1+t_1^2)(1-t_2^2) - x(1+t_1^2)(1+t_2^2) \\ p_2 = 2a_1t_1(1+t_2^2) + 2a_2t_2(1+t_1^2) - y(1+t_1^2)(1+t_2^2) \end{bmatrix} = 0. \quad (21)$$

However the step of clearing the denominators introduces extraneous solutions. By using a Sylvester resultant elimination method [33], one can eliminate t_2 from Eqs. (21) to obtain a single polynomial in the remaining variable t_1 , written as

$$4a_2^2(1+t_1^2)^3((x^2+y^2+2a_1x+a_1^2-a_2^2)t_1^2-4ya_1t_1+x^2+y^2-2a_1x+a_1^2-a_2^2) = 0. \quad (22)$$

This is an 8th order polynomial with the quadratic factor $(1+t_1^2)$ repeated three times. The second quadratic term in (22) gives two assembly modes of the RR dyad. This means that system (21) has eight solutions and six of them are extraneous associated with the cleared denominators $1+t_j^2=0$. To solve system (21) with a homotopy solver, we can use the following set structure for the start system

$$S_1 = \{\{t_1\}^2, \{t_2\}^2\}, S_2 = \{\{t_1\}^2, \{t_2\}^2\}, \quad (23)$$

which gives a GLP Bézout number of 8. By tracking 8 solution paths, six of them lead to extraneous solutions.

To eliminate the extraneous solutions, we augment the system (21) by the following two constraint equations:

$$\begin{bmatrix} p_3 = 1 - l_1(1 + t_1^2) \\ p_4 = 1 - l_2(1 + t_2^2) \end{bmatrix} = 0.$$

The GLP set structure for the constraint polynomials is given as the following

$$S_3 = \{\{t_1\}^2, \{l_1\}\}, S_4 = \{\{t_2\}^2, \{l_2\}\} , \quad (24)$$

which is combined with S_1 and S_2 in (24) to form the augmented set structure $\{S_1, S_2, S_3, S_4\}$. The GLP Bézout number of the augmented set structure is also 8.

By tracking these eight paths with POLSYS_GLP or HOM4PS2, we obtain the two isolated solutions which are what we seek for. The other six paths lead to solutions at infinity. This result shows that the constrained homotopy technique successfully eliminates all six extraneous solutions introduced in the formulation process. It is important to point out that if the constrained homotopy technique is not used, Bertini with regeneration mode also traced 8 solution paths. However, it computed 4 solutions with additional post-processing revealing that 2 of them are extraneous. This is due to the fact that those solutions are isolated points which cannot be eliminated by Bertini.

4.2. Revisit to the nine-point path generation of four-bar linkages

The goal of this problem is to find four-bar linkages whose coupler points trace nine predefined precision points. Many authors have attempted to

solve this problem. Morgan and Wampler [34] reported there are at most 36 non-degenerate solutions to this problem for the case of five precision points. Roth and Freudenstein [21] also using the bootstrap partially solved the problem. Another attempt using Cheater’s Homotopy, a kind of more reliable numerical Continuation than Bootstrap, was employed by [35] which could offer again a partial solution to the problem of this type. Finally Wampler et al. [32] fulfilled the task of solving the problem using the homotopy continuation method. They formulated the problem in isotropic coordinates [36] so as to correctly shrink the number of solutions. By use of symmetry, linearity of some variables and the idea of multi-homogeneous Bézout number they reduced the problem into 12 equations to trace only 143260 paths. After the solution process, in the next step, they used a post-processing to refine all the singular solutions and the solutions which were detected to be degenerate to finally arrive to at most 1442×3 non-degenerate solutions and equivalently 1442 nondegenerate Roberts cognates configurations [37]. In the refine process the underlying criterion to re-track the suspicious endpoints, was based on high condition number and satisfiability of the degeneracy conditions governing the physics of the problem. But in the refine process a special care must be taken to correctly distinguish singular or positive dimensional solutions so as not to miss any nonsingular solutions. This in return needs to track all the paths with a small step length or in the best case to re-track some suspicious endpoints [32]. Beyond the extra hand-crafted computation work, there still exists an ambiguous hesitation whether we found all the nonsingular solutions or not as the result depends on the choice of criterion.

A schematic view of a four-bar with its triangular coupler is shown in Figure 6 (a). Figure 6(b) shows the vector definitions of the corresponding

four-bar following the isotropic formulation from [32]. As may be seen in Figure 6(c), submitting a displacement δ_j , $j = 1, \dots, 8$, P_0 displaces to a new precision point P_j and C and D displace respectively to new associated points namely C' and D' .

From Figure 6, one may consider two independent chains namely, $AP_0P_jD'A$ and $BP_0P_jC'B$. Writing down the equations of these vector loops with $j = 1, \dots, 8$, gives

$$\begin{bmatrix} (x-a)e^{i\lambda_j} = xe^{i\theta_j} + \delta_j - a \\ (y-b)e^{i\mu_j} = ye^{i\theta_j} + \delta_j - b \end{bmatrix}, \quad (25)$$

These sets of equations as was done by [32] can be reduced to a system of 8 equations

$$p_{1-8} : \quad \gamma_j \hat{\gamma}_j + \gamma_j \gamma_j^0 + \hat{\gamma}_j \gamma_j^0 = 0, \quad j = 1, \dots, 8, \quad (26)$$

based on the following determinants,

$$\gamma_j = \begin{vmatrix} (a - \delta_j)\hat{x} & \delta_j(\hat{a} - \hat{x}) + \hat{\delta}_j(a - x) - \delta_j\hat{\delta}_j \\ (b - \delta_j)\hat{y} & \delta_j(\hat{b} - \hat{y}) + \hat{\delta}_j(b - y) - \delta_j\hat{\delta}_j \end{vmatrix}, \quad (27)$$

$$\hat{\gamma}_j = \begin{vmatrix} \delta_j(\hat{a} - \hat{x}) + \hat{\delta}_j(a - x) - \delta_j\hat{\delta}_j & (\hat{a} - \hat{\delta}_j)x \\ \delta_j(\hat{b} - \hat{y}) + \hat{\delta}_j(b - y) - \delta_j\hat{\delta}_j & (\hat{b} - \hat{\delta}_j)y \end{vmatrix}, \quad (28)$$

$$\gamma_j^0 = \begin{vmatrix} (\hat{a} - \hat{\delta}_j)x & (a - \delta_j)\hat{x} \\ (\hat{b} - \hat{\delta}_j)y & (b - \delta_j)\hat{y} \end{vmatrix}, \quad (29)$$

Moreover, in [32] the transformations

$$p_{9-12} : \quad \{n = a\hat{x}, \quad \hat{n} = \hat{a}x, \quad m = b\hat{y}, \quad \hat{m} = \hat{b}y\}, \quad (30)$$

were used to lower multi-homogenous Bézout number. As a result, the 2-homogeneous Bézout number of the new 12 sets of equations associated with

$\{x, \hat{x}, a, \hat{a}, n, \hat{n}\}$ and $\{y, \hat{y}, b, \hat{b}, m, \hat{m}\}$ is 286720. The 10-homogeneous Bézout number is also the same as 286720 but the 2-homogeneous Bézout number due to the brevity is more preferable.

Not all the solutions of this system of 12 equations may define a physically acceptable four-bar linkage. From Figure 6 it is obvious that whenever A and D , D and C , C and B and finally A and B coincide, the ultimate system of polynomials will have degenerate solutions. Furthermore, from the reduction process as is explained in [32], once P_0 coincides either D or C i.e. for the trivial values of γ_j^0 , extraneous solutions will be produced. These degenerate solutions are represented by the polynomials $g_j = 0 (j = 1, \dots, 12)$ written as

$$g_{1-12} : \left\{ \begin{array}{cccccc} a - x, & a - b, & x - y, & b - y, & x, & y, \\ \hat{a} - \hat{x}, & \hat{a} - \hat{b}, & \hat{x} - \hat{y}, & \hat{b} - \hat{y}, & \hat{x}, & \hat{y} \end{array} \right\}, \quad (31)$$

Finally, the constrained homotopy technique suggests the following to augment to the system of equations.

$$\mathcal{G} : p_{13-24} = \begin{bmatrix} 1 - l_1(a - x) \\ 1 - l_2(a - b) \\ 1 - l_3(x - y) \\ 1 - l_4(b - y) \\ 1 - l_5x \\ 1 - l_6y \\ 1 - l_7(\hat{a} - \hat{x}) \\ 1 - l_8(\hat{a} - \hat{b}) \\ 1 - l_9(\hat{x} - \hat{y}) \\ 1 - l_{10}(\hat{b} - \hat{y}) \\ 1 - l_{11}\hat{x} \\ 1 - l_{12}\hat{y} \end{bmatrix} = 0, \quad (32)$$

Augmenting the foregoing equations to the previous 12 equations gives a system of 24 equations in 24 unknowns for which the classical homotopy solvers are now able to simply solve for the nondegenerate solutions only.

Numerical Results

To sample a generic problem, nine generic precision points were defined using random complex numbers as the input data. We used both Bertini (which employs a multi-homogeneous start system) and HOM4PS2 (which employs a polyhedral homotopy) to solve both the original and the augmented systems. In case of HOM4PS2 we did not use the transformation, Eqs. (30), since the mixed volume without the transformation is smaller. To reduce computational burden and increase precision in Bertini runs, we exploited the straight line feature which allows defining intermediate terms in the input data file. As far as computer platform, we used the parallel

version of Bertini on the High Performance Computing facility of the University of Maryland, Baltimore County, while we ran HOM4PS2 on a 64-bit Linux PC. For this reason, we do not report the timing of all runs but rather focus on the solutions yielded.

A summary of the comparison is tabulated in Table 1. The results clearly show the inability of classical homotopy methods to eliminate degenerate or positive dimensional solutions. As it is seen, Bertini in classical mode gives as many as 224639 solutions, of which only 8652 or 3.8% are nondegenerate. In case of HOM4PS2, 13.6% of the total solution obtained are nondegenerate. When the constrained homotopy technique is used, both homotopy solvers only reported the nondegenerate solutions. One should appreciate the proposed approach because if other numerical methods such as Newton-Raphson iteration methods are used to solve a polynomial system like this, there is a very high possibility that the algorithm will converge to one of the degenerate solutions.

By computing a number of randomly generated cases, Bertini with the constrained homotopy technique reported a stable root count of 8652 which is consistent with the result by the regeneration mode [9]. The stable root count is partially attributed to the adaptive precision of arithmetic implemented in Bertini. Please note that HOM4PS2 reported a slightly different root count of 8635 due to numerical errors introduced by insufficient precision of arithmetic. In summary, we confirm that the number of nondegenerate solutions for the nine point path generation of a four-bar is 8652 which is in the form of 1442 triplets with a dual symmetry, i.e. $8652 = 1442 \times 3 \times 2$. This result agrees with that of [32] who obtained the result by retracking suspicious paths with a carefully chosen tolerance.

Discussion

As one can see, our technique is very easy to use for engineers. One can simply append more equations to a polynomial system to exclude unwanted solution, isolated or positive dimensional. This would require no extra step of post-processing the solutions or modifying homotopy codes. Currently there is no homotopy codes which provide this functionality.

Even the augmented system constructed by our technique consists of more equations than the original system, the multi-homogeneous Bézout number, GLP Bézout number and the mixed volume give the same number of paths to be followed. Although it does create bigger matrices in the path tracking process due to the introduction of new variables l , our numerical results show that total amount of time needed to solve the same problem with and without our technique are almost the same. Of course one can minimize the effect on timing by developing a specialized tracker that takes advantage of the special structure of the constraint polynomials.

Besides these, our technique is compatible with Bertini in regeneration mode. Regeneration both with and without our technique gives only the nondegenerate solutions. Compared with the regeneration mode of Bertini, our technique offers more flexibility as the user has the full control on which set of solutions are to be eliminated. For instance, suppose we would like to eliminate only a part of positive dimensional solutions of the nine point path generation problem which are associated with a portion of the total constraints. Then based on our technique we can simply append those corresponding constraint polynomials, but not all the equations in (32), to construct the augmented system. While on the other hand to achieve the same goal with the regeneration mode of Bertini, a post-processing pro-

cedure would be inevitable as it will automatically eliminate all positive dimensional solutions.

Furthermore the solutions obtained with the constrained homotopy technique seem to have a better accuracy than the traditional post-processing approach. This is probably due to the fact that all homotopy codes employ a well developed end-game routine to distinguish between finite and infinite solutions. To show this, we wrote a post-processing routine in C++ code which eliminates all degenerate solutions based on the user-defined tolerance. Please note that this tolerance should be equal to or less than the one employed in the homotopy tracking.

We ran Bertini in the classical mode without using the constraint polynomials to solve sets of generic input data and obtained three sets of solutions (including positive dimensional solutions). We then executed our routine to post-process these solutions with a tolerance of 10^{-10} which was considered as the admissible residual function tolerance employed in the continuation solution step. Table 2 clearly indicates that only checking the solutions against the algebraic equations $g_j(x) = 0$ was not capable of identifying all isolated solutions. This is because the solutions obtained do not have a sufficient precision if the constraint polynomials are not used. This means that without making use of multi-precision arithmetic some of the homotopy paths must be retracked with a tighter tolerance. The simplest and maybe the most effective criterion to identify these suspicious endpoints can be that of [32]. However this step needs a careful numerical attention which in return increases the time of the solution process which on the other hand is totally obviated upon employing the new technique.

5. Summary and Conclusions

In this paper, we proposed a technique called the “Constrained Homotopy” technique, which enables the classical homotopy methods to eliminate unwanted solutions. This technique is very easy to use for engineers as they can simply append more equations to the original polynomial system. It does not require any additional work to the solution set or the homotopy codes such as post-processing, path re-tracking and modifying the homotopy codes. Furthermore users have a full control on which solution sets are to be eliminated or kept. This convenient technique saves engineers much efforts in solving practical problems. We also showed that the extra constraint polynomials do not change the multi-homogeneous and GLP Bézout numbers or the mixed volume of the original system. As a result, solving the augmented system requires almost the same amount of time needed to solve the original problem. Furthermore our numerical experiments show that the solutions obtained with the proposed technique tend to have a better accuracy compared with the solutions when the technique is not used.

Two practical kinematics problems were employed to demonstrate the use of this technique with the classical homotopy method. In the inverse kinematics of an RR dyad problem, we showed that the constrained technique successfully eliminates six isolated extraneous solutions and returns exactly the two useful solutions. In the nine-point path generation of four-bar problem, we showed that the large number of degenerate solutions were directed to the solutions at infinity. As a result, we can accurately report that the number of solutions for this classical problem is 1442 triplets with a dual symmetry. This result is in complete agreement with the previous findings. Compared with the published result obtained by a tedious post-

processing technique, our method offers more user friendliness, reliability and numerical stability.

In addition, we also showed that our result agrees with the result obtained by the regeneration technique which is typically used for identifying positive dimensional solutions. As a matter of fact, our technique is also compatible with regeneration and can be used in parallel with regeneration method where one needs more reliability on the solution process. Compared with the regeneration procedure which automatically eliminates all the positive dimensional solutions, our technique offers more freedom as one can decide either to keep or eliminate the solutions on any of the algebraic sets.

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References

- [1] M. Raghavan, B. Roth, Solving Polynomial systems for the Kinematic Analysis and Synthesis of Mechanisms and Robot Manipulators, ASME, Special 50th Anniversary Design Issue, Journal of Mechanical Design 117 (1995) 71–79.

- [2] C. B. Garcia, W. I. Zangwill, Global Continuation Methods for Finding all Solutions to Polynomial Systems of Equations in N Variables, Center for Math Studies in Business and Economics Report 7755, Univ. of Chicago, Chicago, IL, 1977.
- [3] E. L. Allgower, K. Georg, Introduction to Numerical Continuation Methods, SIAM, Philadelphia, 2003.
- [4] A. P. Morgan, Solving Polynomial Systems Using Continuation for Scientific and Engineering Problems, Prentice-Hall, Englewood Cliffs, NJ, 1987.
- [5] T. Y. Li, Numerical Solution of Polynomial Systems by Homotopy Continuation Methods, Handbook of Numerical Analysis 11 (1) (2003) 209–304.
- [6] A. J. Sommese, C. W. Wampler, The Numerical Solution of Systems of Polynomials arising in Engineering and Science, World Scientific Press, Singapore, 2005.
- [7] B. Huber, B. Sturmfels, A polyhedral method for solving sparse polynomial systems, American Mathematical Society, Mathematics of Computation 64 (212) (1995) 1541–1555, ISSN 0025-5718, doi: <http://dx.doi.org/10.2307/2153370>.
- [8] D. N. Bernshteín, The number of roots of a system of equations, Functional Analysis and Appl. 3 (1975) 183–185.
- [9] J. D. Hauenstein, A. J. Sommese, C. W. Wampler, Regeneration Homotopies for Solving Systems of Polynomials, to appear, Mathematics of Computation, AMS, 2010 .

- [10] L. T. Watson, M. Sosonkina, R. C. Melville, A. P. Morgan, H. F. Walker, Algorithm 777: HomPack90: a suite of Fortran 90 codes for globally convergent homotopy algorithms, *ACM Trans. Math. Softw.* 23 (4) (1997) 514–549, ISSN 0098-3500, doi: <http://doi.acm.org/10.1145/279232.279235>.
- [11] S. M. Wise, A. J. Sommese, L. T. Watson, Algorithm 801: POLSYS_PLP: a partitioned linear product homotopy code for solving polynomial systems of equations, *ACM Trans. Math. Softw.* 26 (1) (2000) 176–200, ISSN 0098-3500, doi: <http://doi.acm.org/10.1145/347837.347885>.
- [12] H.-J. Su, J. M. McCarthy, M. Sosonkina, L. T. Watson, Algorithm 857: POLSYS_GLP—a parallel general linear product homotopy code for solving polynomial systems of equations, *ACM Trans. Math. Softw.* 32 (4) (2006) 561–579, ISSN 0098-3500.
- [13] T. L. Lee, T. Y. Li, C. H. Tsai, HOM4PS-2.0: a software package for solving polynomial systems by the polyhedral homotopy continuation method, *Computing* 83 (2-3) (2008) 109–133.
- [14] T. Y. Li, C. H. Tsai, HOM4PS-2.0para: Parallelization of HOM4PS-2.0 for Solving Polynomial Systems, *Parallel Computing* 35 (4) (2009) 226–238.
- [15] T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, T. Mizutani, PHoM: a Polyhedral Homotopy Continuation Method for Polynomial Systems, *Computing* 73 (1) (2004) 57–77, ISSN 0010-485X, doi: <http://dx.doi.org/10.1007/s00607-003-0032-4>.

- [16] D. J. Bates, J. D. Hauenstein, A. J. Sommese, C. W. Wampler, Software for numerical algebraic geometry: a paradigm and progress towards its implementation, *Software for Algebraic Geometry, The IMA Volumes in Mathematics and its Applications* 148 (2008) 1–14.
- [17] A. J. Sommese, C. W. Wampler, *Numerical Algebraic Geometry, Lectures in Applied Mathematics* 32 (1996) 749–763.
- [18] A. J. Sommese, J. Verschelde, Numerical Homotopies to Compute Generic Points on Positive Dimensional Algebraic Set, *Journal of Complexity* 16 (2000) 572–602.
- [19] A. J. Sommese, J. Verschelde, C. W. Wampler, Numerical Decomposition of the Solution sets of Polynomial Systems into Irreducible Components, *SIAM, J. Numerical Analysis* 38 (6) (2001) 2022–2046.
- [20] J. Verschelde, Algorithm 795: PHCpack: a general-purpose solver for polynomial systems by homotopy continuation, *ACM Trans. Math. Softw.* 25 (2) (1999) 251–276, ISSN 0098-3500, doi: <http://doi.acm.org/10.1145/317275.317286>.
- [21] B. Roth, F. Freudenstein, Synthesis of Path-Generating Mechanisms by Numerical Methods, *ASME, Journal of Engineering for Industry* 85 (B3) (1963) 298–306.
- [22] L. W. Tsai, A. P. Morgan, Solving the Kinematics of the most General Six- and Five-Degree of Freedom Manipulators by Continuation Methods, *ASME, Journal of Mechanisms, Transmissions and Automations in Design* 107 (1985) 189–200.

- [23] H.-Y. Lee, C.-G. Liang, Displacement analysis of the general spatial 7-link 7R mechanism, *Mechanism and Machine Theory* 23 (3) (1988) 219 – 226.
- [24] M. Raghavan, B. Roth, Kinematic analysis of the 6R manipulator of general geometry, in: *In International Symposium on Robotics Research*, MIT Press, 314–320, 1989.
- [25] M. Raghavan, The Stewart Platform of General Geometry has 40 Configurations, *ASME, Journal of Mechanical Design* 115 (1993) 277–282.
- [26] C. W. Wampler, Forward displacement analysis of general six-in-parallel sps (Stewart) platform manipulators using soma coordinates, *Mechanism and Machine Theory* 31 (3) (1996) 331–337.
- [27] M. L. Husty, An algorithm for solving the direct kinematics of general Stewart-Gough platforms, *Mechanism and Machine Theory* 31 (4) (1996) 365–379.
- [28] C. Innocenti, Forward Kinematics in Polynomial Form of the General Stewart Platform, *ASME, Journal of Mechanical Design* 123 (2) (2001) 254–260.
- [29] C. W. Wampler, A. P. Morgan, A. J. Sommese, Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics, *ASME, Journal of Mechanical Design* 112 (1) (1990) 59–68.
- [30] A. J. Sommese, J. Verschelde, C. W. Wampler, Advances in Polynomial Continuation for Solving Problems in Kinematics, *Journal of Mechanical Design* 126 (2) (2004) 262–268.

- [31] J. Duffy, C. Crane, A displacement analysis of the general spatial 7-link, 7R mechanism, *Mechanism and Machine Theory* 15 (3) (1980) 153–169.
- [32] C. W. Wampler, A. P. Morgan, A. J. Sommese, Complete Solution of the Nine-Point Path Synthesis Problem for Four-Bar Linkages, ASME, *Journal of Mechanical Design* 114 (1) (1992) 153–159.
- [33] D. Cox, J. Little, D. O’Shea, *Ideals, Varieties and Algorithms*, Third Edition, Springer, 2007.
- [34] A. P. Morgan, C. W. Wampler, Solving a Planar Four-Bar Design Problem using Continuation, ASME, *Journal of Mechanical Design* 112 (1990) 544–550.
- [35] L. W. Tsai, J. J. Lu, Coupler-Point-Curve Synthesis Using Homotopy Methods, ASME, *Journal of Mechanical Design* 112 (3) (1990) 384–389.
- [36] C. W. Wampler, Isotropic coordinates, circularity and Bézout numbers: Planar Kinematics from a new perspective, *Proceedings of the ASME Design Engineering Technical Conference, (CDROM), Irvine, CA*, 18–22, 1996.
- [37] S. Roberts, On Three-Bar Motion in Plane Space, *Proceedings of the London Mathematical Society III(1869-71)* (1875) 286–319.

Appendix

For $P(x) = (p_1(x), \dots, p_n(x))$, $x \in \mathbb{C}^n$, let S_i be the support of $p_i(x)$ for $i = 1, \dots, n$. Let $C = (\{a_1, b_1\}, \dots, \{a_n, b_n\})$ be a cell of type $(1, 1, \dots, 1)$ of a *fine mixed subdivision* Γ of $\mathcal{S} = (S_1, \dots, S_n)$. (We shall simply call cells of such type *mixed cells*.) Huber and Sturmfels [7] showed that the mixed volume of $P(x)$ can be calculated by

$$\begin{aligned} MV(P) &= \sum_{\substack{\text{All mixed cells} \\ C \in \Gamma}} Vol(conv(C)) \\ &= \sum_{\substack{\text{All mixed cells} \\ C \in \Gamma}} \left| \det \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix} \right| \end{aligned}$$

where $conv(C)$ is the convex hull of the mixed cell C . And the start system for the polyhedral homotopy is given by

$$Q(x) : \quad q_i(x) = \sum_{\alpha \in S_i \cup \{0\}} c_{i,\alpha} x^\alpha, \quad i = 1, \dots, n \quad (33)$$

where $S_i \in \mathbb{Z}^n$ is the set of exponent vectors of monomials in $p_i(x)$ and $c_{i,\alpha}$ are coefficients chosen generically. Also see [5] for a comprehensive introduction on the polyhedral homotopy method.

Theorem 3. *The mixed volume of the augmented system $\hat{P}(x, l)$ equals the mixed volume of the original system $P(x)$, i.e. $MV(\hat{P}) = MV(P)$.*

PROOF. For the augmented system $\hat{P}(x, l)$ with support $\hat{\mathcal{S}} = (\hat{S}_1, \dots, \hat{S}_{n+m})$, elements in \hat{S}_i for $i = 1, \dots, n$ are in the form $(a_1^{(i)}, \dots, a_n^{(i)}, 0, \dots, 0)$ where $(a_1^{(i)}, \dots, a_n^{(i)}) \in S_i$, while for $n + j$, $j = 1, \dots, m$, \hat{S}_{n+j} consists of elements

of the form

$$\left(\overbrace{*, \dots, *}^n, \overbrace{0, \dots, 0}^j, 1, 0, \dots, 0 \right)$$

together with $(0, \dots, 0) \in \mathbb{R}^{n+m}$. Let

$$\hat{C} = (\{\hat{a}_1, \hat{b}_1\}, \dots, \{\hat{a}_n, \hat{b}_n\}, \{\hat{a}_{n+1}, \hat{b}_{n+1}\}, \dots, \{\hat{a}_{n+m}, \hat{b}_{n+m}\})$$

be a mixed cell of a fine mixed subdivision of $\hat{\mathcal{S}} = (\hat{S}_1, \dots, \hat{S}_{n+m})$ where for $i = 1, \dots, n$, $\hat{a}_i = (a_i, 0)$ and $\hat{b}_i = (b_i, 0)$ with $\{a_i, b_i\} \subset S_i$ and $0 = (0, \dots, 0) \in \mathbb{R}^m$. It is clear that $(\{a_1, b_1\}, \dots, \{a_n, b_n\})$ forms a mixed cell of a *induced* fine mixed subdivision of $\mathcal{S} = (S_1, \dots, S_n)$. Now, for volume of $\text{conv}(\hat{C})$ to be nonzero, for each $j = 1, \dots, m$, one of $\{\hat{a}_{n+j}, \hat{b}_{n+j}\}$ must be $(0, \dots, 0) \in \mathbb{R}^{n+m}$. Without loss of generality, suppose $\hat{b}_{n+j} = 0 \in \mathbb{R}^{n+m}$ for all $j = 1, \dots, m$. So,

$$\begin{aligned} \text{Vol}(\text{conv}(\hat{C})) &= \left| \det \begin{pmatrix} \hat{a}_1 - \hat{b}_1 \\ \vdots \\ \hat{a}_{n+m} - \hat{b}_{n+m} \end{pmatrix} \right| = \left| \det \begin{pmatrix} a_1 - b_1, & 0, \dots, 0 \\ \vdots & \vdots \\ a_n - b_n, & 0, \dots, 0 \\ * \dots *, & 1, \dots, 0 \\ \vdots & \ddots \\ * \dots *, & 0, \dots, 1 \end{pmatrix} \right| \\ &= \left| \det \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix} \right|. \end{aligned}$$

By the aforementioned calculation, the mixed volume of the augmented system $\hat{P}(x, l)$ is therefore equal to the mixed volume of the system $P(x)$.

□

Table 1: Number of solutions for example (3) with/without our technique.

Method		without the technique		with the technique	
		# of Paths Tracked	# of Solutions	# of Paths Tracked	# of Solutions
Bertini	Classical	286720	224639*	286720	8652
	Regeneration	146348*	8652	250172*	8652
HOM4PS2		103783	63363*	103783	8635*

*This number slightly changes depending on the input data.

Table 2: Comparison of constrained homotopy technique with post-processing.

Post-Process		# of solutions: Before	# of solutions: After
Set 1	Unaugmented	224636	9068
	Augmented	8652	8652
Set 2	Unaugmented	224635	8854
	Augmented	8652	8652
Set 3	Unaugmented	224638	9105
	Augmented	8652	8652

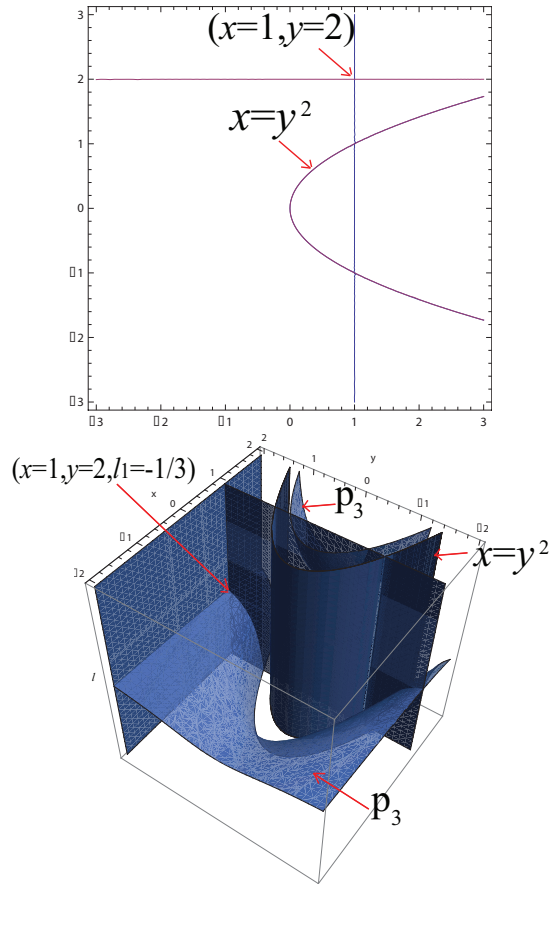


Figure 1: (a) The algebraic curves of a system of two polynomials and (b) Three-dimensional view of the augmented polynomial system.

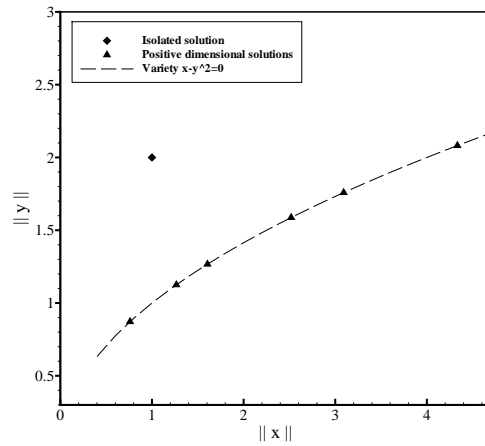


Figure 2: The solutions obtained without using the constraint polynomial.

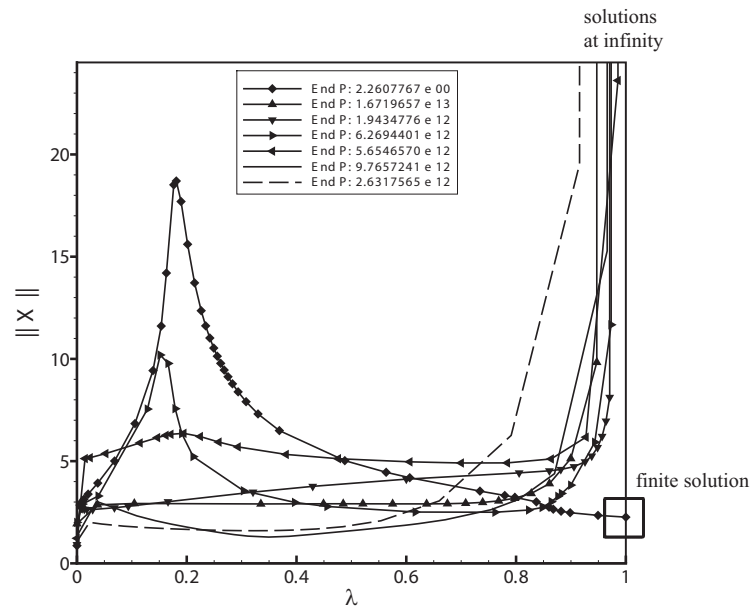


Figure 3: Euclidean norm of homotopy paths obtained by the codes POLSYS_GLP.

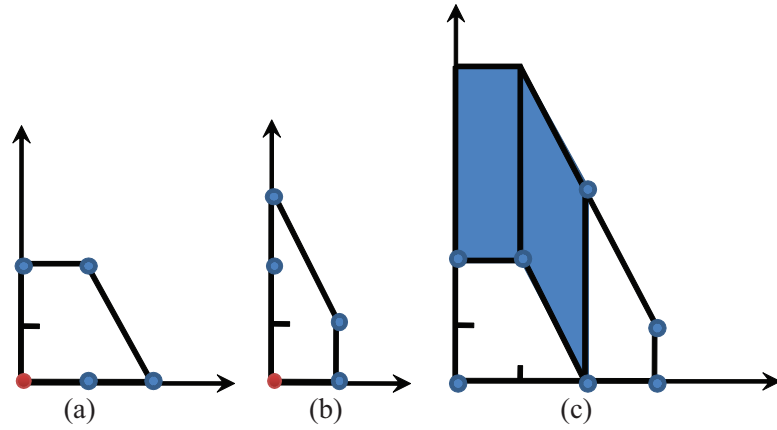


Figure 4: The stable mixed volume of the original polynomials.

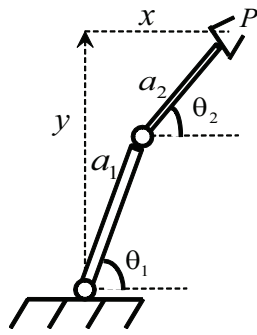


Figure 5: A schematic view of a robot arm with two rigid links and two pin joints.

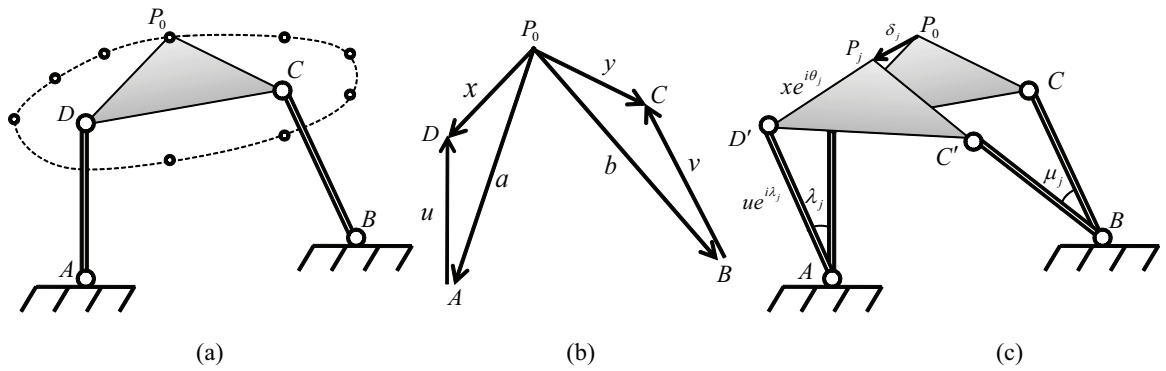


Figure 6: (a) A schematic view of a four-bar and its coupler curve, (b) the vector representation of its geometry (c) the displaced configuration.