

# A Complete Homotopy Solution to the Eight-Point Path Generation of a Slider-Crank Four-Bar Linkage

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*In this paper, we study the synthesis of a slider-crank four-bar linkage whose coupler point traces a set of predefined task points. We report that there are at most 558 slider-crank four-bars in cognate pairs passing through any eight specified task points. To the authors' knowledge, this is the first time to report this solution count. The problem is formulated for up to eight precision points in polynomial equations. The classical elimination methods are used to reduce the formulation to a system of seven 6th degree polynomials. To sift out the degenerate solutions, a recently invented constrained homotopy technique is employed. Essentially the degenerate solutions are mapped to solutions at infinity of the augmented system. This eliminates the traditional post-processing procedure which can be tedious and requires extra programming work. To obtain solutions to the augmented system, we propose a solution process based on the classical homotopy and the secant homotopy methods. At last two numerical examples are provided to verify our formulation and solution process. In second example, we obtained six slider-crank linkages without a branch or an order defect. This is partially attributed to an innovative strategy of choosing design points on a fourth degree polynomial curve.*

## 1 Introduction

The goal of path generation problems is to find linkages whose coupler points pass through a given set of points. In the past decades, extensive research on this topic have been carried out. Most of the works have focused on the path generation of planar four-bar linkages which can exactly pass through at most nine precision points. Early investigations were devoted to the cases of four or five precision points but with different formulations and approaches. Among those, Shigley and Uicker [1] approached the problem using graphical methods, which however reported to be limited up to six

precision points [2]. A second approach was optimization problem as was conducted by Angeles et al. [3]. Freudenstein and Sandor [4], Morgan and Wampler [5], and Subbian and Flugrad [6] also employed the analytical approach which was shown to be effective for even more number of precision points.

Basically for the case of nine points but a special case of geared five-bar, first attempt appears to be conducted by Roth and Freudenstein [7] in which using their new developed approach, so called Bootstrap but still a kind of numerical continuation method, they provided a partial solution to the problem. Tsai and Lu [8] with using Cheater's Homotopy, a numerical method even more reliable than Bootstrap, did not accomplish the complete solution. Finally by classical and numerical reduction of the problem, using the numerical continuation method, Wampler et al. [9] achieved the complete solutions to the nine-point path generation of four-bar linkages. Their criteria to eliminate degenerate solutions, solutions which do not correspond to acceptable physical dimensions, have been to re-track the suspicious endpoints with tighter tolerances and postprocessing the remaining solutions based on the degeneracy conditions governing the physics of the problem. As a result, considering a dual symmetry and moreover a three-way symmetry due to Robert's cognates [10], they reported 1442 generic configuration triplets as the upper bound to the number of solutions to this problem.

Very recently this problem was revisited by Tari et al. [11] in which they developed a Constrained Homotopy technique for numerical continuation method to eliminate unwanted solutions, possibly degenerate or extraneous, from the polynomials arising in kinematics problems. To be precise, the Constrained Homotopy technique applies more reliability to the extraneous and the positive dimensional solutions elimination by forcing the unwanted solutions to infinite solutions of a system of a higher dimension which therefore can be taken care of using variable homogenization when continuation methods are used. The new results

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are in complete agreement with the previous findings.

The polynomial homotopy or continuation method has been proved to be robust and effective enough for solving kinematics problems. As one of the early applications of the method to kinematics problems, Raghavan [12] was the first who numerically proved there are at most 40 solutions to the Stewart platform of general geometry. Tsai and Morgan [13] also generated complete solution to the kinematics problems of the most general six- and five-degree of freedom manipulators by using numerical continuation method.

Recently enormous advances [14, 15] in both aspects of homotopy theory and computational codes have been made. In particular, extensive works have been done to adapt the homotopy method to identify positive dimensional solutions of a square or a non-square polynomial system. This is generally done by a regeneration technique [16] which solves the polynomial system equation by equation. Slicing the given system of polynomials each time with a general linear space of complementary dimension leads to witness solutions on positive dimensional algebraic sets. After years of development, many homotopy packages are freely available. The most recent coding efforts include POLSYS\_GLP [17], Bertini [18] and HOM4PS2 [19, 20]. Bertini besides supporting a user-defined homotopy, provides the regeneration technique for finding isolated and positive dimensional solutions and is capable of dealing with non-square systems of polynomials [21–23]. HOM4PS2 is a polyhedral homotopy solver which in general is a better choice for solving sparse systems.

In practice, relatively less work has been done on the path generation of planar slider-crank four-bar linkages. As one of the few works [24] presents an  $m$ -homogenization formulation of the path generation of slider-crank four-bars using real variables. However their formulation does not offer degenerate solutions elimination and maximum number of slider-crank four-bar configurations that pass through any set of generally defined precision points. In this paper, we study designing slider-crank four-bar linkages to generate a path through some given task points. First, we give an overview of the studied mechanism in the following section. In section 3, using isotropic coordinates [25] the problem was mathematically modeled which illustrates the fact that the coupler point of any general slider-crank four-bar linkage can pass through at most eight given task points, then in section 4, the resulting set of algebraic equations which design parameters should satisfy have been reduced using classical elimination techniques to finally arrive to a set of seven polynomials of sixth degree. However after the classical reduction, using the Constrained Homotopy technique [11], all the degenerate solutions, the solutions which make a configuration physically unacceptable, have been eliminated. Meanwhile a secant homotopy scheme to reduce the online computational burden, which is important for daily based design practices is offered. Finally numerical examples and conclusions are given.

## 2 The Slider-Crank Four-Bar

Figure 1 shows the geometry of a general slider-crank four-bar with three pin joints  $A, B, C$ . And the point  $B$  is sliding in the direction given by the slope angle  $\beta$ . The point  $P$  is a coupler point. Let us denote the vectors  $PA, PC, PB$  by complex variables  $a, x$  and  $y$  respectively. And we also use complex number  $u = x - a$  to denote the vector  $AC$ . To fully determine a slider-crank four-bar, one has to determine variables  $(a, x, y, \beta)$ , total of seven independent scalar parameters.

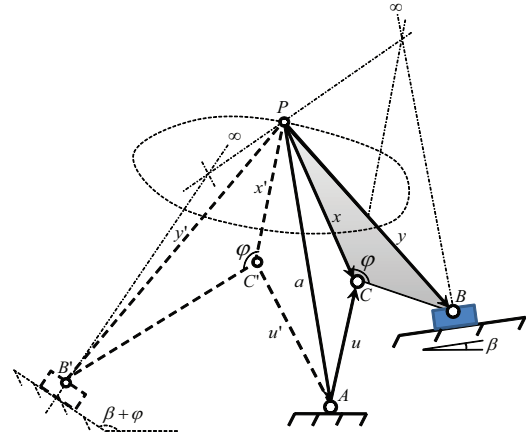


Fig. 1. Slider-crank four-bar cognates.

For any slider-crank four-bar, there exists a unique cognate linkage [10] which traces the same coupler curve. One may construct the cognate of a slider-crank as the following. First construct a parallelogram  $ACPC'$  based on the triangle  $ACP$  to determine point  $C'$ . Then construct the triangle  $PC'B'$  similar to the triangle  $BCP$  to determine point  $B'$ . And the slider angle  $\beta' = \beta + \varphi$ , where  $\varphi$  defines the angle between vectors  $CP$  and  $CB$  as shown in Figure 1. These determine the cognate linkage  $APC'B'$  which shares points  $A$  and  $P$  with the original linkage. Essentially for any slider-crank with the parameters  $(a, x, y, \beta)$  we define its cognate with

$$(a' = a, x' = a - x, y' = \frac{x - a}{x - y}y, \beta' = \beta + \varphi).$$

## 3 Problem Formulation

The goal of the synthesis problem in this paper is to design a planar slider-crank linkage whose coupler point  $P$  traces a set of predefined task points  $P_j (j = 0, 1, \dots, n)$ . See Figure 2(a). Without loss of generality, we place the local coordinate system at the first precision point  $P_0 = (0, 0)$ . As described in the previous section, there are seven independent parameters to determine in designing a slider-crank. Hence we can specify up to eight precision points (including the origin  $P_0$ ), i.e.  $n \leq 7$ , for the slider-crank four-bar path generation problem.

Figure 2(b) shows the mechanism together with the configuration when its coupler point is displaced from  $P_0$  to  $P_j$ .

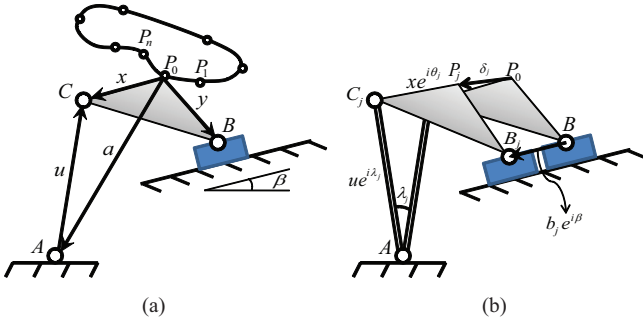


Fig. 2. (a) A schematic view of a slider-crank four-bar and the vector representation of its geometry along with a typical coupler curve (b) the displaced configuration.

Considering two independent vector loops,  $P_0P_jC_jACP_0$  and  $P_0P_jB_jBP_0$  yields the following two sets of complex equations

$$\begin{aligned} f_1 : & \left[ \begin{aligned} (x-a)e^{i\lambda_j} &= xe^{i\theta_j} - a + \delta_j \\ ye^{i\theta_j} &= b_j e^{i\beta} + y - \delta_j \end{aligned} \right], j = 1, \dots, n, \quad (1) \end{aligned}$$

where  $i = \sqrt{-1}$  is the imaginary unit number,  $b_j$  are the real variables denoting the magnitudes of the slider's displacements,  $\theta_j$  and  $\lambda_j$  are respectively the angular displacements of the coupler triangle and the link AC. And  $\delta_j = P_j - P_0$  are the displacements of the coupler point.

It is worth mentioning right at this point that if a slider-crank four-bar linkage with the parameters  $\{a, x, y, \beta, \lambda_j, \theta_j, b_j\}$  satisfies Eq.(1), then indirectly so does its cognate given by  $\{a, a-x, \frac{x-a}{x-y}y, \beta + \varphi, \theta_j, \lambda_j, \frac{\|x\|b_j}{\|x-y\|}\}$  as it satisfies  $yf_1 + xf_2$  and  $f_2$ . On the other hand, it is easy to see that the cognate of the cognate of such a mechanism is itself. These justify that cognates of slider-crank four-bars appear in pairs and are included in our formulation.

The conjugate of Eq. (1) yields another two sets of complex equations as follows

$$\left[ \begin{aligned} (x^* - a^*)e^{-i\lambda_j} &= x^*e^{-i\theta_j} - a^* + \delta_j^* \\ y^*e^{-i\theta_j} &= b_j e^{-i\beta} + y^* - \delta_j^* \end{aligned} \right], j = 1, \dots, n, \quad (2)$$

where asterisk represents the complex conjugation.

Let us define complex variables  $\hat{a} = a^*$ ,  $\hat{x} = x^*$  and  $\hat{y} = y^*$ . We also redefine complex variables  $\hat{\lambda}_j = e^{i\lambda_j}$ ,  $\hat{\lambda}_j = e^{-i\lambda_j}$ ,  $\hat{\theta}_j = e^{i\theta_j}$ ,  $\hat{\theta}_j = e^{-i\theta_j}$ ,  $\hat{\beta} = e^{i\beta}$  and  $\hat{\beta} = e^{-i\beta}$ . As a result, Eqs. (1,2) can be rewritten as

$$\begin{aligned} f_1 : & \left[ \begin{aligned} (x-a)\lambda_j &= x\theta_j - a + \delta_j \\ y\theta_j &= b_j\hat{\beta} + y - \delta_j \end{aligned} \right] \\ f_2 : & \left[ \begin{aligned} (\hat{x}-\hat{a})\hat{\lambda}_j &= \hat{x}\hat{\theta}_j - \hat{a} + \delta_j^* \\ \hat{y}\hat{\theta}_j &= b_j\hat{\beta} + \hat{y} - \delta_j^* \end{aligned} \right], j = 1, \dots, n, \quad (3) \end{aligned}$$

Since we treat  $\{\hat{\beta}, \hat{\lambda}_j, \hat{\lambda}_j, \hat{\theta}_j, \hat{\theta}_j\}$  as independent variables,

they must also satisfy the following identity constraints

$$\begin{aligned} f_5 : & \left[ \begin{aligned} \lambda_j \hat{\lambda}_j &= 1 \\ \theta_j \hat{\theta}_j &= 1 \\ \hat{\beta} &= 1 \end{aligned} \right], j = 1, \dots, n. \quad (4) \end{aligned}$$

Please note that Eqs.(3,4) form a system of  $6n + 1$  polynomial equations in  $5n + 8$  unknowns, namely  $\{a, \hat{a}, x, \hat{x}, y, \hat{y}, \beta, \hat{\beta}, \lambda_j, \hat{\lambda}_j, \theta_j, \hat{\theta}_j, b_j\}$ . When the number of equations equals the number of unknowns, we obtain the maximum case of  $n = 7$  as shown previously. For this case ( $n = 7$ ), we have to solve a system of 43 quadratic polynomial equations in 43 variables. The total-degree of this system is  $2^{43}$ , almost  $8.8 \times 10^{12}$ , solution of which appears to be infeasible even with the current state of the art computers. Therefore we will reduce the complexity of this formulation in the following section.

#### 4 Formulation Reduction

To eliminate  $\lambda_j, \hat{\lambda}_j$ , we multiply  $f_1$  and  $f_3$  in (3) and substitute identity equations  $f_5, f_6$  from (4). On the other hand, to eliminate  $\theta_j$  and  $\hat{\theta}_j$ , we first solve  $f_2$  and  $f_4$  for  $\theta_j$  and  $\hat{\theta}_j$  and substitute them in the result of the previous step and clear the denominators. Finally in the last step, we multiply  $f_2$  and  $f_4$  and substitute the identity equations  $f_6, f_7$ . These procedures lead us to a new system

$$\begin{aligned} \Phi_j : & \left[ \begin{aligned} \phi_{j1}b_j + \phi_{j0} &= 0 \\ b_j^2 + b_j\psi_{j1} + \psi_{j0} &= 0 \\ \hat{\beta}\hat{\beta} &= 1 \end{aligned} \right], j = 1, \dots, n. \quad (5) \end{aligned}$$

where the coefficients are given by

$$\begin{aligned} \phi_{j0} &= x\hat{y}\hat{\alpha}_j\eta_j + \hat{x}y\alpha_j\hat{\eta}_j - (a\hat{\chi} + \hat{a}\chi + \delta_j\delta_j^*)y\hat{y} \\ \phi_{j1} &= x\hat{y}\hat{\alpha}_j\hat{\beta} + \hat{x}y\alpha_j\hat{\beta} \\ \psi_{j0} &= \eta_j\hat{\eta}_j - y\hat{y} \\ \psi_{j1} &= \eta_j\hat{\beta} + \hat{\eta}_j\hat{\beta} \end{aligned}$$

where  $\alpha_j = a - \delta_j$ ,  $\hat{\alpha}_j = \hat{a} - \delta_j^*$ ,  $\chi_j = x - \delta_j$ ,  $\hat{\chi}_j = \hat{x} - \delta_j^*$ ,  $\eta_j = y - \delta_j$  and  $\hat{\eta}_j = \hat{y} - \delta_j^*$ .

Considering the most general case i.e.  $n = 7$ , system (5) consists of seven 5-th degree polynomials, seven cubics and one quadric therefore summing to 15 polynomials with the total degree of  $2 \times 3^7 \times 5^7 \approx 3.42 \times 10^8$  which still seems considerably large to be practically solved.

Let us eliminate  $b_j$  by taking the Sylvester resultant [26] of  $\Phi_j$  and  $\Psi_j$ , calculated as

$$\text{Res}(\Phi_j, \Psi_j, b_j) = \begin{vmatrix} \phi_{j1} & 0 & 1 \\ \phi_{j0} & \phi_{j1} & \psi_{j1} \\ 0 & \phi_{j0} & \psi_{j0} \end{vmatrix}, j = 1, \dots, n. \quad (6)$$

After expanding the above determinant and substituting the identity  $\beta = 1/\beta$ , we factored out the term  $y\hat{y}$  and cleared the denominator  $\beta$ . We found the degree of  $\beta$  in the resultant (6) to be of four and with only even power of  $\beta$  appearing. This was expected because  $\beta$  and  $-\beta$  define the same sliding direction as shown in Figure 1.

To remove this symmetry and reduce the degree of  $\beta$ , we use the transformation  $\beta^2 \leftarrow \beta$ . Then we also define  $\hat{m} = \beta\hat{y}$  to lower the multi-homogeneous Bézout number of the system but still retaining the same number of equations. For the case  $n = 7$ , we obtain the final set of polynomial equations

$$P(\mathbf{z}) : p_j = x\hat{\alpha}_j s_{j1} + s_{j2} + \hat{x}\alpha_j s_{j3}, \quad j = 1, \dots, 7 \quad (7)$$

where,

$$\begin{aligned} s_{j1} &= x\hat{\alpha}_j \hat{m}^2 + s_{j4} s_{j7} \\ s_{j2} &= (s_{j6} - \hat{m} y s_{j5}) s_{j5} + 2(\delta_j \hat{m} + \delta_j^* \beta \eta_j) x \hat{\alpha}_j \hat{\alpha}_j \\ s_{j3} &= \hat{x} y^2 \alpha_j + \eta_j (x \hat{\alpha}_j \eta_j - y s_{j5}) \\ s_{j4} &= \hat{x} \alpha_j s_{j7} - \hat{m} s_{j5} \\ s_{j5} &= a \hat{\chi}_j + \hat{a} \chi_j + \delta_j \delta_j^* \\ s_{j6} &= \hat{m} x \hat{\alpha}_j \eta_j + \hat{x} y \alpha_j s_{j7} \\ s_{j7} &= \hat{m} - \delta_j^* \beta \end{aligned}$$

The above formulation is coded in the input file feeding to the homotopy code Bertini which takes advantage of straight line programming to speed up computation and increase precision while in case of HOM4PS2,  $P(\mathbf{z})$  is expanded.

System (7) consists of seven 6th degree polynomials in seven variables,  $\mathbf{z} = \{a, \hat{a}, x, \hat{x}, y, \hat{m}, \beta\}$  with the total degree of  $6^7 = 279,936$ , meaning 7.5 times less than  $8^7 = 2,097,152$  that of [24], therefore resulting in a significant reduction. We call system  $P(\mathbf{z})$  the ‘‘unaugmented’’ system.

## 5 The Augmented Polynomial System

Not all the mathematical solutions to the unaugmented system  $P(\mathbf{z}) = 0$  in Eqs. (7), give us a practical slider-crank linkage. From Figure 2(a), it is obvious that once  $A$  and  $C$  and moreover  $B$  and  $C$  coincide the resulting configurations would not be physically acceptable as they will not represent slider-crank four-bars anymore. Furthermore, if either  $B$  or  $C$  coincides  $P_0$  the resulting configurations are not general since they trace the precision points which are located only on a line or a circle respectively. We call such solutions the degenerate solutions and the conditions which define these the degeneracy conditions. These cases bring up four degeneracy conditions but upon working in complex domain so do their conjugations. They may be summarized by the following polynomials.

$$g_{1-8}(\mathbf{z}) : \left\{ \begin{array}{l} x - a, \quad x - y, \quad x, \quad y \\ \hat{x} - \hat{a}, \quad \beta \hat{x} - \hat{m}, \quad \hat{x}, \quad \hat{m} \end{array} \right\} \quad (8)$$

We are interested in finding the nondegenerate solutions which are the solutions of  $P(\mathbf{z}) = 0$  excluding those satisfying any of  $g_{1-8}(\mathbf{z}) = 0$ .

Traditionally to sift out extraneous or degenerate solutions, one has to post-process solutions [9] by checking a list of degeneracy conditions which may require extra programming and re-tracking suspicious paths with a tighter tolerance. Very recently Tari et al. [11] proposed a ‘‘Constrained Homotopy’’ technique which enables classical continuation methods to eliminate unwanted degenerate solutions possibly isolated or positive dimensional solutions from the polynomial systems. Essentially this technique forces the unwanted solutions to the solutions at infinity of a polynomial system of a higher dimension which can hence be sifted out by existing homotopy codes. We apply this technique for our problem to eliminate all degenerate solutions without using a post-processing procedure.

According to the Constrained Homotopy technique [11], we augment the original polynomial system  $P(\mathbf{z})$  by ‘‘constrained polynomials’’  $\mathcal{G}(\mathbf{z}, \mathbf{l})$  written as

$$\mathcal{G}(\mathbf{z}, \mathbf{l}) : p_{8-15} = \begin{bmatrix} 1 - l_1(x - a) \\ 1 - l_2(x - y) \\ 1 - l_3 x \\ 1 - l_4 y \\ 1 - l_5(\hat{x} - \hat{a}) \\ 1 - l_6(\beta \hat{x} - \hat{m}) \\ 1 - l_7 \hat{x} \\ 1 - l_8 \hat{m} \end{bmatrix} \quad (9)$$

where  $\mathbf{l} = (l_1, \dots, l_8)$  are new complex variables. It is obvious that all the degenerate solutions which satisfy  $g_{1-8}$  are mapped to the infinite solutions of the ‘‘augmented’’ system  $\hat{P}(\mathbf{z}, \mathbf{l})$ ,

$$\hat{P}(\mathbf{z}, \mathbf{l}) : \begin{bmatrix} P(\mathbf{z}) = 0 \\ \mathcal{G}(\mathbf{z}, \mathbf{l}) = 0 \end{bmatrix}. \quad (10)$$

Tari [11] et al. showed that all the nondegenerate solutions of  $P(\mathbf{z})$  are included in the solutions of  $\hat{P}(\mathbf{z}, \mathbf{l})$  neglecting the variables  $\mathbf{l}$ . Furthermore, they showed that the number of homotopy paths of the augmented system equals to that of the unaugmented system, meaning that the computational amount is not significantly increased due to the use of the augmented system. In what follows, we show how to use the homotopy method to solve the augmented system  $\hat{P}(\mathbf{z}, \mathbf{l}) = 0$  and obtain the solutions,  $(\mathbf{z}, \mathbf{l})$ .

## 6 The Solution Process

Numerical homotopy continuation method due to its capability of systematically solving relatively large polynomial systems is employed as the method to solve the augmented system  $\hat{P}$  which was obtained by three major analytical reduction steps as elaborated in the previous section. The polynomial homotopy or continuation method which appears to

Table 1. Number of solutions to unaugmented and augmented systems respectively  $P(\mathbf{z}) = 0$  and  $\hat{P}(\mathbf{z}, \mathbf{l}) = 0$ .

Method		$p_{1-7}$		$p_{1-15}$	
		# of Paths Tracked	# of Solutions	# of Paths Tracked	# of Solutions
Bertini	Classical	26880	14582*	26880	558
	Regeneration	14576*	558	19036*	558
HOM4PS2		5632	2348**	5632	558***

\*This number slightly changes,  $\pm 0.1\%$ , depending on the input data.

\*\*This number slightly changes,  $\pm 1\%$ , depending on the input data.

\*\*\*This number slightly changes,  $-1\%$ , depending on the input data.

be first proposed by Garcia and Zangwill [27], starts at roots of a trivial *start system*  $Q(\mathbf{z})$  and traces roots along the so-called *homotopy paths* with the real variable  $t$  as the start system  $Q(\mathbf{z})$  from  $t = 0$  is continuously transformed into the target system  $P(\mathbf{z})$  at  $t = 1$  using the following homotopy.

$$H(\mathbf{z}, t) = (1-t)\gamma Q(\mathbf{z}) + tP(\mathbf{z}) = 0 \quad \text{with generic } \gamma \in \mathbb{C} \setminus \{0\}.$$

For our problem, the target system is the augmented system  $\hat{P}(\mathbf{z}, \mathbf{l}) = 0$  as shown in (10) with  $(\mathbf{z}, \mathbf{l})$  being the variables to be solved.

### 6.1 The maximum root count

To generate a generic sample in the problem family, we define  $\delta_i$ ,  $i = 1, \dots, 7$  with randomly chosen complex numbers with unit 2-norms, then both unaugmented and augmented systems of  $P$  respectively  $p_{1-7}$  and  $p_{1-15}$  are solved using two publicly available polynomial solver packages namely Bertini [28] and HOM4PS2 [19].

For Bertini, we use an  $m$ -homogeneous start system with the following partition:

$$S_1 = \{a, \hat{a}\}, S_2 = \{x, \hat{x}\}, S_3 = \{y, \hat{m}, \beta\}, S_4 = \{l_1, \dots, l_8\},$$

which gives a 4-homogenous Bézout number of 26880. See [29] for several concrete examples of  $m$ -homogeneous Bézout number calculation.

It is important to note that, based on the following transformations

$$\mathbf{n} = a\hat{x}, \hat{\mathbf{n}} = \hat{a}x$$

for the newly constructed nine equations, the 3-homogeneous Bézout number corresponding to

$$\{y, \hat{m}, \beta\}, \{a, \hat{a}, x, \hat{x}, \mathbf{n}, \hat{\mathbf{n}}\}, \{l_1, \dots, l_8\},$$

is 17920 which is more desirable for the offline solution step and also consistent with the homotopy root count based on

a real variable formulation given in [24]. But considering the fact that, the offline step is to be performed once and the online computation part, discussed later, is to be repeated for practical problems, we shall not use the transformations to keep our polynomial system with fewer equations. Doing so we avoid extra programming step to reformatting the solution list needed to initiate the online computation step.

On the other hand, the stable mixed volume of our augmented system reported by HOM4PS2 is as low as 5632. This was expected since the polyhedral root count takes a great advantage of sparsity of the monomials appearing in the polynomials.

The results of the different runs are summarized in Table 1. Augmented polynomials unlike the unaugmented ones lead us only to nondegenerate solutions counting to 558 or 279 pairs of slider-crank four-bars. This stable root count reported by Bertini is independent of the input data, while HOM4PS2 slightly depends on the choices of the input data. Regeneration mode of Bertini also reports a stable root count of 558, independent of the Constrained Homotopy technique. However, the counter example given in [11] has justified that regeneration method without the Constrained Homotopy technique is incapable of eliminating extraneous isolated solutions of with multiplicity greater than one. The last observation only guarantees that our formulation procedure produces no extraneous solution as the stand alone regeneration method gives the same number of solutions as that of the Constrained Homotopy technique. See [30] for an example of an inappropriate formulation procedure which introduces extraneous solutions.

As a result, we conclude that there are in general at most 558 nondegenerate solutions or a total of 279 pairs of slider-crank four-bar cognates, whose coupler points may pass through any eight general predefined task points. As far as the authors' knowledge, this is the first time that the maximum number of solutions for the eight-point path generation of the slider-crank linkage is reported.

### 6.2 The secant homotopy continuation

As may be seen, only 558 paths out of 26880 homotopy paths leave us finite solutions. Computation for the other paths lead us to infinite solutions. This is a waste of com-

puter time and makes the online computational infeasible. Fortunately this step is only a one time computation and for subsequent problems we can use the solutions of the generic problem obtained above to construct a start system and only track the exact number of solution paths i.e. 558 paths. To achieve this we use “secant homotopy” [14] which often offers improvements in the speed and simplicity of the program. The secant homotopy is given by

$$H(\mathbf{z}, t) = (1 - t)\gamma P_1(\mathbf{z}) + tP_2(\mathbf{z}) = 0,$$

where  $P_1(\mathbf{z})$  is a polynomial system already solved and  $P_2(\mathbf{z})$  is a new polynomial system of the same structure but with different coefficients to be solved. For our problem,  $P_1(\mathbf{z})$  is the generic problem solved in the previous section. By taking advantage of the solutions of this generic problem, the secant homotopy only tracks 558 paths to obtain the solutions to  $P_2(\mathbf{z})$ . However, devising a carefully designed homotopy such as parameter homotopy enables one to track even 279 paths when cognates are ignored.

As a matter of fact, in order to construct and use the secant homotopy it should be first justified that it leads to all the nonsingular solutions i.e. 558 solutions. According to [14], with  $\lambda_1$  and  $\lambda_2$  as random complex numbers and  $P_A(\mathbf{z})$  and  $P_B(\mathbf{z})$  two generic problems, we solved  $\lambda_1 P_A(\mathbf{z}) + \lambda_2 P_B(\mathbf{z})$  using regeneration method which resulted in 558 nonsingular solutions. As a result, this allows us to use the secant homotopy for any practical problem of this type.

### 6.3 The solution framework

Figure 3 shows the framework of solving the path generation problem of the slider-crank linkage. The solution process consists of two parts, the offline part and the online part. The offline part is to solve a generic problem  $P_1(\mathbf{z}) = 0$  using a full scale homotopy such as  $m$ -Homogeneous or polyhedral. This requires tracking 26880 paths when using an  $m$ -Homogeneous start system or 5632 paths when using a polyhedral homotopy solver. The 558 solutions are saved to a data file. This part takes about a few hours or almost ten minutes on a single processor PC once  $m$ -Homogeneous or polyhedral start systems are to be used, respectively. In the online part, we use any set of design position specified by the designers to construct a new target system  $P_2(\mathbf{z})$ . The start system is constructed by using the already solved polynomial system  $P_1(\mathbf{z})$  and its solutions. We then use a secant homotopy routine provided by Bertini to solve  $P_2(\mathbf{z}) = 0$  by tracking only 558 paths. Note that the online computation only takes a few minutes on a PC.

In addition, we have posted the solution file and the input data file for setting secant homotopy online for free download<sup>1</sup>. Designers can simplify or modify the data for the design points and obtain the complete solution set by running Bertini program.

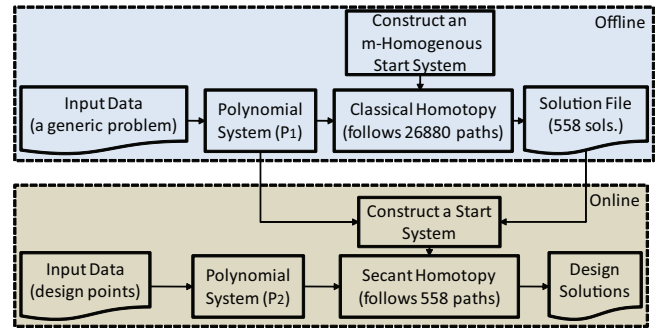


Fig. 3. The solution process using the homotopy continuation method.

## 7 Numerical Examples

In this section, we demonstrate our solution process with two numerical examples.

### 7.1 Example 1

In this example, we generate the design points by finding 8 points on the coupler curve of the slider-crank defined by the following parameters:

$$\begin{aligned} a &= -0.10883628135782344 - 0.39151204568758424i \\ x &= -0.01807417216889669 - 0.34953300884005322i \\ y &= -0.11962372344482726 - 0.25316035384842301i \\ \beta &= -1.49298260011097472 \end{aligned}$$

We chose eight points on the resulting coupler curve, as given in Table 2. The mechanism and its coupler curve along with the chosen task points are depicted in Figure 4. It is obvious

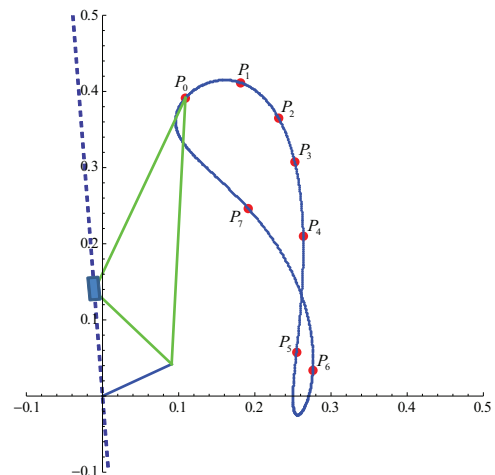


Fig. 4. A schematic view of a slider-crank four-bar and its coupler curve based on the values given in Table 2.

that this mechanism should be included in the final solutions to the system  $P(\mathbf{z})$ .

<sup>1</sup><http://www.umbc.edu/engineering/me/vrml/research/slidersynthesis/>

Table 2. The task points for the example 1.

$P_0$	$0.1088362813578234 + 0.3915120456875842 i$
$P_1$	$0.1815133698757787 + 0.4114601071246998 i$
$P_2$	$0.2308533661531308 + 0.3655170210841163 i$
$P_3$	$0.2523798766179371 + 0.3073014212734279 i$
$P_4$	$0.2634884480780613 + 0.2094601474681307 i$
$P_5$	$0.2546251612800458 + 0.0578903785139394 i$
$P_6$	$0.2760920032162055 + 0.0334110572087386 i$
$P_7$	$0.1908832819710217 + 0.2467168400729561 i$

With the eight coupler points given in Table 2, we defined the target system  $P_2$  which is solved using the secant homotopy. Using a single processor of our PC, Bertini took almost six minutes to solve the problem with following 558 paths. However, ignoring the cognates, i.e. following 279 paths, would save considerable time to fulfill the computation. Among the obtained solutions only those whose  $a$  and  $\hat{a}$ ,  $x$  and  $\hat{x}$  and finally  $y$  and  $\hat{m}/\beta$  are conjugate define the actual and useful physical solutions. This is normally done by extra programming to scan over all the solutions against a tolerance which in our case is set to the allowable residual function evaluation employed in the continuation step namely,  $10^{-10}$ . With this done, 62 solutions or 31 pairs of linkages including the original one shown in Figure 4 are found. Another solution is shown in the Figure 5. Please note that this is a slider-rocker four-bar since the coupler curve is not a closed curve. Unfortunately all other 29 linkage pairs including the cognates of the mechanisms shown in Figures 4 and 5 have a branch or an order defect.

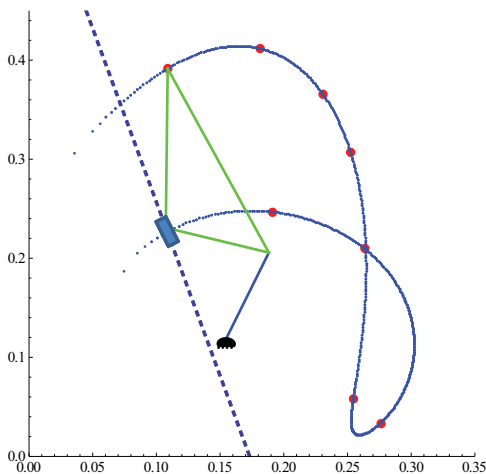


Fig. 5. A slider-rocker four-bar and its coupler curve as a solution to example 1.

## 7.2 Example 2

As one may see, most of the solutions obtained in the first example have a defect. These solutions are not the solutions engineers want. One way to think about this is that each design point can be on either branch of the slider-crank linkage. Thus the chance to obtain a solution without a branch defect would be roughly at least as low as  $1/2^8 = 1/256$ . In other words, solving for slider-crank four-bars for any arbitrary known coupler curve most likely results in branch defect solutions. To increase the chance to obtain useful solutions, we propose an approximate solution framework described as follows. It is known that a general slider four-bar linkage generates a fourth degree coupler curve [31]. Therefore for any set of user specified design points  $P_j (j = 0, 1, \dots, 7)$ , we choose a suitable extra auxiliary point to fit a fourth degree polynomial curve  $f(x, y) = a_0 + \sum_{i=1}^4 (a_i x + b_i y)^i$  to the given points. We then pick different but more appropriate design points on this curve and obtain solutions by following the aforementioned solution process.

To test our framework, we chose the following fourth degree polynomial:

$$f(x, y) = 10^{-8}(5x + 7y)^4 + 10^{-6}(-5x + y)^3 + 10^{-4}(5x - 3y)^2 + 10^{-2}(4x + 8y + 7)$$

which has been rationalized for the sake of simplification. We then picked eight points on this curve, as tabulated in Table 3. The curve  $f(x, y) = 0$  and the picked design points are shown in Figure 6.

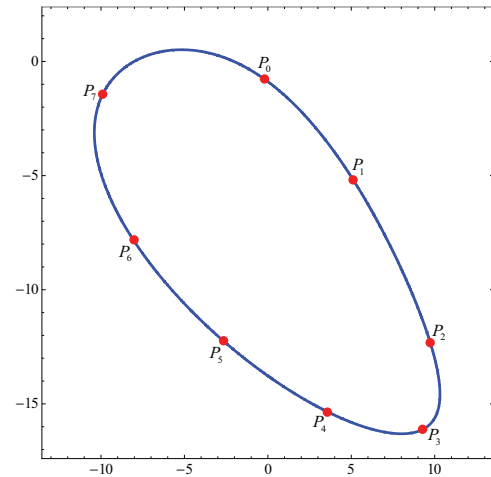


Fig. 6. Graph of  $f(x, y)$  and the sample points on it for example 2.

For the task points given in Table 3, we obtained 47 pairs of linkages. Six of them do not have a branch defect which are shown in Figure 7.

## 8 Conclusions

This paper proposed a complete homotopy solution to the eight-point path generation of a slider-crank four-bar. Af-

Table 3. Sample design points picked on  $f(x,y) = 0$ .

$P_0$	-0.1999868147380189-0.7774410840074841 $i$
$P_1$	5.1245774000261757-5.1959727169682870 $i$
$P_2$	9.7299633625197707-12.312972716968287 $i$
$P_3$	9.2901491055454559-16.115605885925880 $i$
$P_4$	3.5690151005022996-15.342989181549531 $i$
$P_5$	-2.6791006032175608-12.232378702832928 $i$
$P_6$	-8.0449561552805926-7.8333787028329281 $i$
$P_7$	-9.8916000362665973-1.4263891815495310 $i$

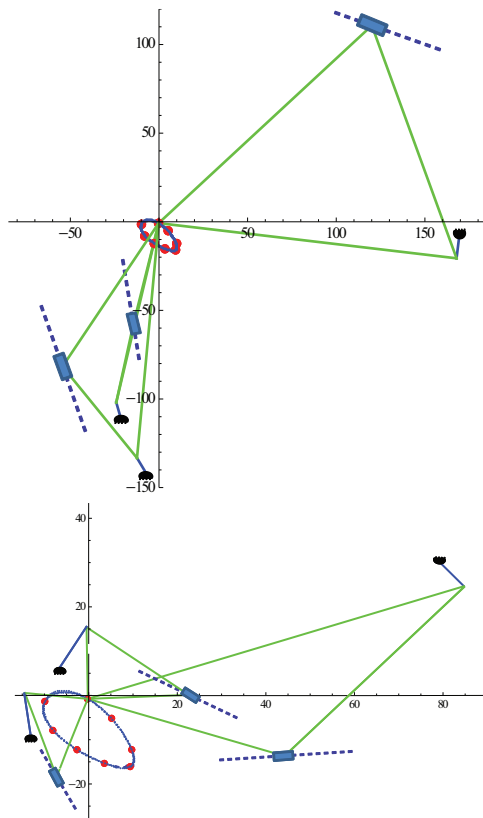


Fig. 7. Six useful solutions for example 2.

ter a few classical reduction steps, the problem is formulated into a set of seven polynomial equations of sixth degree. To eliminate solutions corresponding to degenerate linkages, we augmented the system by extra constrained polynomials. After solving several generic samples in the problem family, we conclude that there are at most 588 or 279 pairs of cognate slider-crank four-bar linkages passing through any eight arbitrary task points. A solution framework based on the secant homotopy procedure is proposed to solve practical design problems in a daily basis. To facilitate the use of this solution process in solving practical synthesis problems, we posted the input data files online for free download. This allows design engineers solve linkage synthesis problems online.

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