

4.4 DIFFERENTIAL FORMS ON SURFACES

(1)

DEFS

- ① A 0-form on a surface M - function $f: M \rightarrow \mathbb{R}$
- ② A 1-form on M is $w_p: T_p M \xrightarrow{\text{choice}} \mathbb{R} \quad \forall p \in M$.
- ③ A 2-form η on M is choice of

$$\eta_p: T_p M \times T_p M \xrightarrow{\text{choice}} \mathbb{R} \quad \text{so that}$$

$$\eta_p(\vec{v}, \vec{w}) = -\eta_p(\vec{w}, \vec{v}) \quad \forall \vec{v}, \vec{w} \in T_p M.$$

EXS

- ① If $M \subset \mathbb{R}^3$ and ω is a p -form on \mathbb{R}^3 Then ω defines a p -form on M ; by restriction.
- ② γ_A = Area Form on M is 2-form on M .

COORD EXPNS FOR ω ON M Let ω be p -form on M

LET $\vec{x}_\alpha: U_\alpha \rightarrow M$ be coord chart for M .

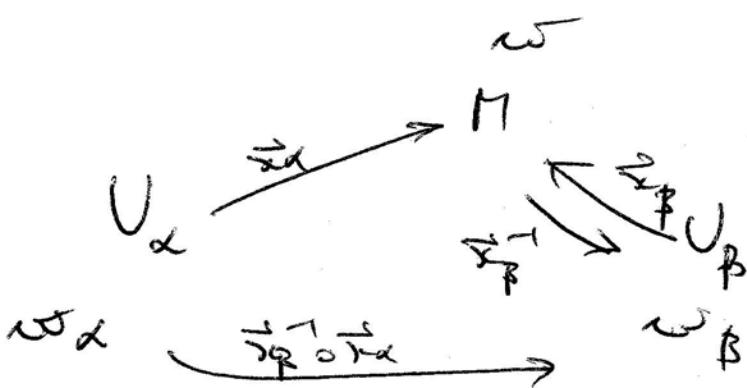
$$\begin{array}{ccc} \vec{x}_\alpha & : & U_\alpha \rightarrow M \\ \downarrow & & \downarrow \\ \mathbb{R}^2 & & \mathbb{R}^3 \end{array}$$

Then Define $\omega_\alpha = (\vec{x}_\alpha)^* \omega$ p -form on U_α by

$$\omega_\alpha(\vec{v}_1, \dots, \vec{v}_p) = \omega((\vec{x}_\alpha)_+^*(\vec{v}_1), \dots, (\vec{x}_\alpha)_+^*(\vec{v}_p)).$$

ω_α is coord exp for ω in chart \tilde{x}_α . ②

Notes
CLAIM I



$$\boxed{\omega = (\tilde{x}_\beta^{-1} \circ \tilde{x}_\alpha)^* \omega_\beta. \quad \text{④}}$$

P.F.

~~$\omega_\alpha(\tilde{v}_1 - \tilde{v}_p)$~~

$$(\tilde{x}_\beta^{-1} \circ \tilde{x}_\alpha)^*(\omega_\beta)(\tilde{v}_1 - \tilde{v}_p) = \omega_\beta(\tilde{x}_{\beta*}^{-1}\tilde{x}_\alpha(\tilde{v}_1), \dots, \tilde{x}_{\beta*}^{-1}\tilde{x}_\alpha(\tilde{v}_p)) \\ = \omega(\tilde{x}_{\beta*}^{-1}\tilde{x}_{\beta*}^{-1}\tilde{x}_{\alpha*}(\tilde{v}_1), \dots,)$$

$$= \omega(\tilde{x}_{\alpha*}(\tilde{v}_1) - \tilde{x}_{\alpha*}(\tilde{v}_p)) = \omega_\alpha(\tilde{v}_1 - \tilde{v}_p)$$

D

CLAIM II

Suppose we have $\{\omega_\alpha\}_{\alpha \in I}$ on U_α so that ④ holds

Then Define

$$\omega(\tilde{v}_1 - \tilde{v}_p) = \omega_\alpha((\tilde{x}_\alpha^{-1})_*(\tilde{v}_1) - (\tilde{x}_\alpha^{-1})_*(\tilde{v}_p)) \text{ for } \alpha \in I$$

Then ω is a well-defined p-form on M .

DO CHAIN RULE HERE.

DEF Let ω be a p -form on M . ③
Define $d\omega$ to be the $(p+1)$ -form on M ~~given~~
whose local coordinates are given by

$$(d\omega)_\alpha \overset{\text{def}}{=} \star$$

$$(d\omega)_\alpha = d(\omega_\alpha)$$

\uparrow \uparrow
 $d \text{ on } M$ $d \text{ on } \mathbb{R}^2$

NOTE If ω is a p -form on M that is ~~not~~ zero of η on \mathbb{R}^3 then $d\omega = (d\eta)_M$.
PF $d\omega$ is well-defined

NB $(d\omega)_\alpha = (\tilde{\tau}_\beta^{-1} \circ \tilde{\tau}_\alpha)^*(d\omega)_\beta$ ↗ KNOW
 $\omega_\alpha = F_{\alpha\beta}^* \omega_\beta$

Well

Let $F_{\alpha\beta} = \tilde{\tau}_\beta^{-1} \circ \tilde{\tau}_\alpha : U_\alpha \xrightarrow{\mathbb{R}^2} U_\beta \xrightarrow{\mathbb{R}^2}$

$$\begin{aligned} (F_{\alpha\beta})^*(d\omega)_\beta &= (F_{\alpha\beta})^* d(\omega_\beta) \\ &= d(F_{\alpha\beta})^*(\omega_\beta) \quad \text{CHOOSE} \\ &= d(\omega_\alpha) = (d\omega)_\alpha. \quad \text{as req'd.} \end{aligned}$$

(4)

CHAIN RULE

IF $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and ω is a p-form on \mathbb{R}^2

THEN

$$d(F^*\omega) = F^*(d\omega)$$

PF Do case $p=0$ only

$$\omega \rightsquigarrow f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$F^*\omega \rightsquigarrow F^*f = f \circ F: \mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

$$d(F^*\omega)(\vec{v}) = d(f \circ F)(\vec{v})$$

$$= \nabla(f \circ F) \cdot \vec{v}$$

$$= (f \circ F)_*(\vec{v})$$

$$\stackrel{\text{OR}}{=} (f_* \circ F_*)(\vec{v}) = f_*(F_*(\vec{v}))$$

~~$= \nabla f(F_p) \cdot F_*(\vec{v}) = df_{F_p}(F_*(\vec{v}))$~~

$$= \nabla f(F_p) \cdot F_*(\vec{v}) = df_{F_p}(F_*(\vec{v}))$$

$$= F^*(df)(\vec{v})$$

$$= F^*(d\omega)(\vec{v})$$

4.6 FTC

(1)

OUTLINE OF PROOF OF FTC (Revised Notes)

FTC

$$\int_M dw = \int_M w$$

w a ~~if~~ $(p-1)$ -form
~~M~~ ~~(p+1)-d~~ p-dimel

STEP 1 LOCALIZE

IDEA

Reduce to case where w is supported on a coord patch for M .

Outline

Write $M = \bigcup_{\alpha \in I} V_\alpha$ $V_\alpha = \text{Im } (\tilde{x}_\alpha : D_\alpha \rightarrow \mathbb{R}^n)$

Let $\{\varphi_\alpha : M \rightarrow \mathbb{R}, \alpha \in I\}$ be a partition of unity for covering $M = \bigcup_{\alpha \in I} V_\alpha$.

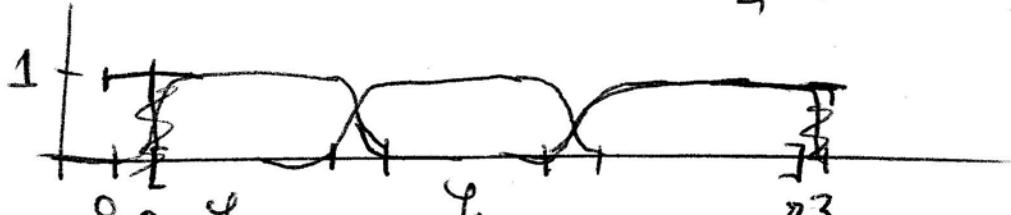
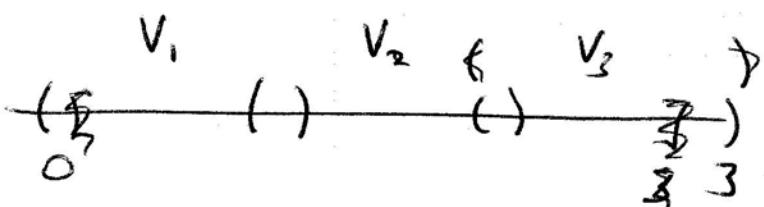
This means

(a) $0 \leq \varphi_\alpha \leq 1$

(b) $\sum_{\alpha \in I} \varphi_\alpha = 1$

(c) $\varphi_\alpha = 0$ on $M \cap V_\beta$. We say $\text{supp}(\varphi_\alpha) \subset V_\alpha$

E.g. $I = \{0, 3\}$



Let $w_\alpha = \varphi_\alpha w$. (2)

ASSUME

$$\int_M dw = \int_M w \quad \text{where } \operatorname{spt}(w) \subset \tilde{\alpha}(\mathbb{D})$$

So we have

$$\int_M dw_\alpha = \int_M w_\alpha$$

NOW

$$w = \sum_\alpha \varphi_\alpha w_\alpha = \sum_\alpha w_\alpha$$

$$dw = \sum_\alpha dw_\alpha$$

So for any w

$$\begin{aligned} \int_M dw &= \int_M \sum_\alpha dw_\alpha = \sum_\alpha \int_M dw_\alpha = \sum_\alpha \int_M w_\alpha \\ &= \int_M \sum_\alpha w_\alpha = \int_M w. \end{aligned}$$

□

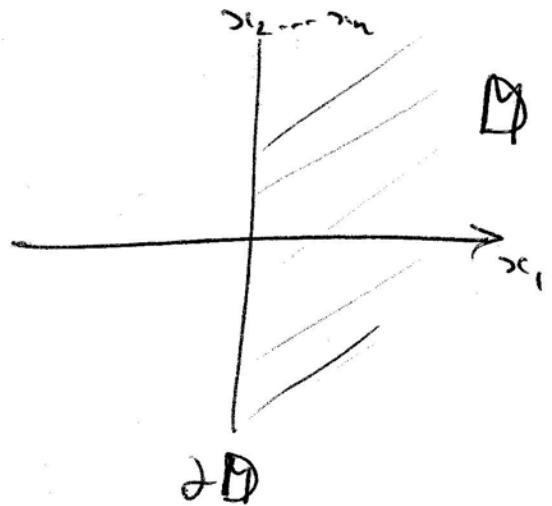
STEP II Reduce to standard model

(3)

Standard Model

$$D = \{ x \in \mathbb{R}^P / x_1 > 0 \}$$

$$\partial D = \{ x \in \mathbb{R}^P / x_1 = 0 \}$$



We can use a coord patch

$$\vec{x}: D \rightarrow M \subset \mathbb{R}^N$$

$$\vec{x}(\partial D) \subseteq \partial M$$

Then for $\text{sopt}(\omega) \subset \vec{x}(D)$:

$$\begin{aligned} \int_M d\omega &= \int_{\vec{x}(D)} d\omega = \int_D \vec{x}^* d\omega \\ &= \int_D d \vec{x}^* \omega \quad \text{as } d\vec{x}^* = \vec{x}^* d \\ &\qquad \qquad \qquad \text{CRWE} \end{aligned}$$

$$\begin{aligned} \text{std} &= \int_{\partial D} \vec{x}^* \omega \\ \text{Model} & \end{aligned}$$

$$= \int_M \omega$$

STEP III (PF FOR STANDARD MODE)

(4)

[STOKES' THM]

IDEAT FROM CASE P ~~2~~ $D = \{(x,y) \in \mathbb{R}^2 / x > 0\} \quad \partial D = \{x=0\}$

By linearity of \int_D we just need to consider 2 cases.

$$\textcircled{1} \quad \omega = \psi(x,y) dx$$

$$\textcircled{2} \quad \psi \omega = \psi(x,y) dy$$

$$\textcircled{1} \quad \omega = \psi dx$$

$$d\omega = \frac{\partial \psi}{\partial x} dx \wedge dx + \frac{\partial \psi}{\partial y} dy \wedge dx = -\frac{\partial \psi}{\partial y} dx \wedge dy$$

$$\begin{aligned} \int_D d\omega &= \int_{x=0}^{\infty} \left(\int_{y=-\infty}^{\infty} \frac{\partial \psi}{\partial y} dy \right) dx \\ &= \int_0^{\infty} [\psi(x, \infty) - \psi(x, -\infty)] dx = 0 \end{aligned}$$

as ψ has compact support

$$\text{and } \int_{\partial D} \omega = \int_{\partial D} \psi dx = \int_{-\infty}^{\infty} \psi(\mathbf{0}, t) dx(\vec{j}) dt = 0$$

$$\begin{aligned} \int_{\partial D} \omega &= \int_{-\infty}^{\infty} \omega_{x(t)}(\alpha'(t)) dt \\ &\stackrel{\alpha(t) = x_j}{=} \int_{-\infty}^{\infty} \omega_{x(t)}(\alpha'(t)) dt \end{aligned}$$

(5)

$$\textcircled{2} \quad w = \Psi dy$$

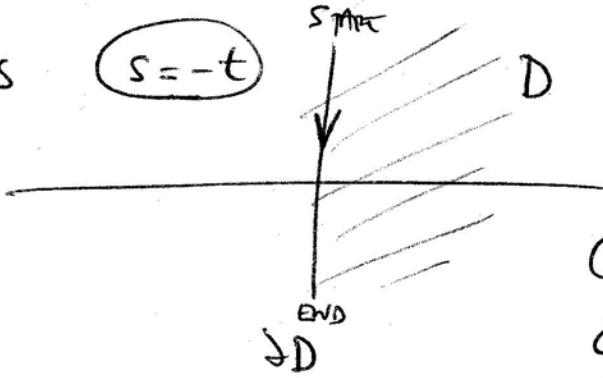
$$dw = \frac{\partial \Psi}{\partial x} dx dy \quad \text{as } \Psi(0, y) = 0$$

$$\int_D dw = \int_{y=-\infty}^{\infty} \left(\int_{x=0}^{\infty} \frac{\partial \Psi}{\partial x} dx \right) dy \stackrel{\text{FTC}}{=} \int_{-\infty}^{\infty} -\Psi(0, y) dy$$

$$\int_D w = - \int_{-\infty}^{\infty} \Psi(0, \bar{s}) dt \quad \text{using } \alpha(t) = -t \rightarrow$$

$$= \int_{+\infty}^{-\infty} \Psi(0, s) ds \quad \text{--- } s = -t$$

$$= - \int_{-\infty}^{\infty} \Psi(0, s) ds$$



Greens Thm
Orient

$$\leq \int_D dw \quad \longrightarrow \quad 0$$