

#1 CALCULUS ON EUCLIDEAN SPACE

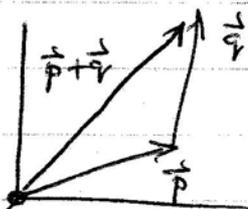
1.1 3D EUCLIDEAN SPACE, \mathbb{R}^3

• POINT OF \mathbb{R}^3 IS $\vec{p} = (p_1, p_2, p_3)$

$\vec{p} \leftrightarrow$ BOLD p
IN BOOK

• \mathbb{R}^3 IS A ~~3D~~ VECTOR SPACE in that

ADDITION If $\vec{p}, \vec{q} \in \mathbb{R}^3$ then $\vec{p} + \vec{q} \in \mathbb{R}^3$



$$\vec{p} + \vec{q} = (p_1 + q_1, p_2 + q_2, p_3 + q_3)$$

SCALAR MULTIPLICATION

If $\vec{p} \in \mathbb{R}^3, a \in \mathbb{R}, a\vec{p} \in \mathbb{R}^3$



ORIGIN

$\exists \vec{0} \in \mathbb{R}^3 : \vec{p} + \vec{0} = \vec{p} \quad \forall \vec{p} \in \mathbb{R}^3$

\mathbb{R}^3 IS A ~~3D~~ VECTOR SPACE of dimension 3 with standard basis

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For any $\vec{p} \in \mathbb{R}^3$: $\vec{p} = p_1 \vec{e}_1 + p_2 \vec{e}_2 + p_3 \vec{e}_3$
 p_1, p_2, p_3 are called COORDINATES of \vec{p} .

We can regard the coordinates as FUNCTIONS on \mathbb{R}^3 .

DEF NATURAL COORD FUNCTIONS $x : \mathbb{R}^3 \rightarrow \mathbb{R}$ by $x(\vec{p}) = p_1$
 $y : \mathbb{R}^3 \rightarrow \mathbb{R}$ $y(\vec{p}) = p_2$
 $z : \mathbb{R}^3 \rightarrow \mathbb{R}$ $z(\vec{p}) = p_3$.

THE FUNCTIONS
ie $f(x,y,z) = x$

Alternate Notation $x_1 = x, x_2 = y, x_3 = z$.

FUNCTIONS ON \mathbb{R}^3 $f : \mathbb{R}^3 \rightarrow \mathbb{R}$

SUBTLE DISTINCTION

EX $f = x^2 + y^2 + \sin z$. is an equation ~~between~~ ^{INVOLVING} FUNCTIONS.

whereas $f(\vec{p}) \oplus (x(\vec{p}))^2 + (y(\vec{p}))^2 + \sin(z(\vec{p}))$

is an equation ~~between~~ ^{INVOLVING} real numbers

In Calculus

We often rewrite \oplus as

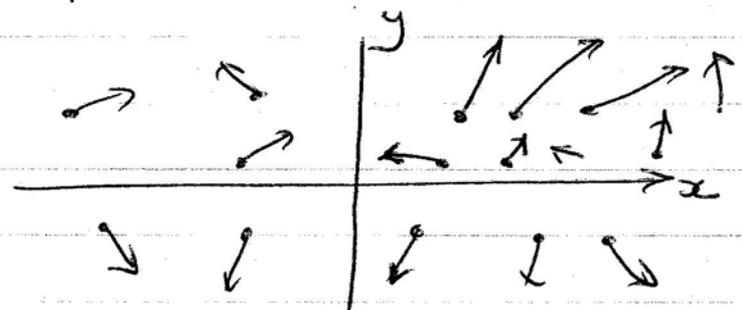
$$f(x,y,z) = x^2 + y^2 + \sin z$$

where now think of x,y,z as numbers rather than functions.

1.2 TANGENT VECTORS

MATH 251, 17.1 VECTOR FIELDS IN PLANE, \mathbb{R}^2 .

A vector field on \mathbb{R}^2 is the assignment of a vector \vec{v}_p to each point p of the plane

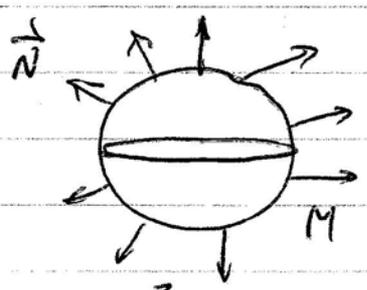


Vector fields are widely used in physics + engineering to model velocity, acceleration, force, etc.

~~We will work with~~

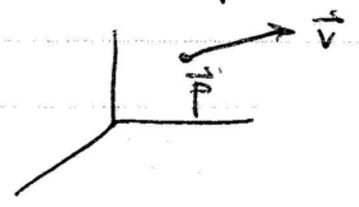
MATH 423 we study

- Tangent vector fields \vec{T} to curves + surfaces
- Normal vector fields \vec{N} to surfaces
- These VFs are only defined on curve/surface not on all of \mathbb{R}^3 .



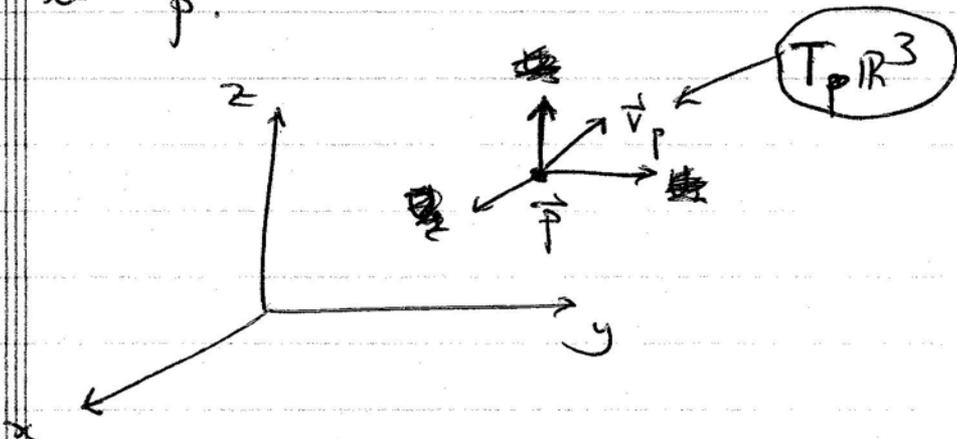
DEF A TANGENT VECTOR \vec{v}_p to \mathbb{R}^3 consists of two points of \mathbb{R}^3 :

- Point of Application p
- Vector part \vec{v}



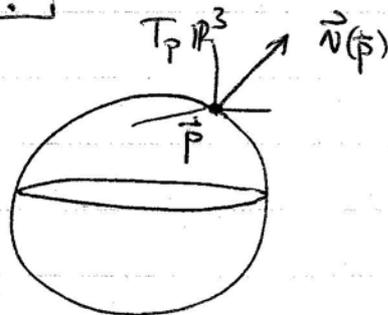
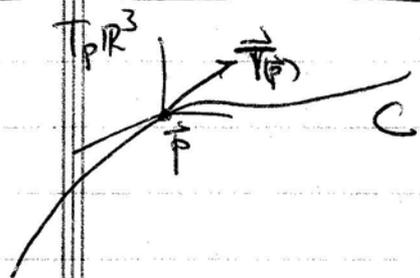
\vec{v} is just a "VECTOR at p "

DEF Let $\vec{p} \in \mathbb{R}^3$. The TANGENT SPACE to \mathbb{R}^3 at \vec{p} is the set $T_p \mathbb{R}^3$ of all tangent vectors \vec{v}_p whose pt of application is \vec{p} .



* \mathbb{R}^3 has a DIFFERENT Tangent Space at each of its pts

WHY ALL THESE TANGENT SPACES?



$$\vec{T}(\vec{p}) \in T_p \mathbb{R}^3$$

$$\vec{N}(\vec{p}) \in T_p \mathbb{R}^3$$

The tangent vector $\vec{T}(\vec{p})$ to a curve C at \vec{p} is really a VECTOR AT \vec{p} , i.e. $\vec{T}(\vec{p}) \in T_p \mathbb{R}^3$!

Of course $T_p \mathbb{R}^3$ is a 3D Vector Space which we can identify with \mathbb{R}^3 by choosing bases for translating pt of appn \vec{p} to origin of \mathbb{R}^3 .

DEF A VECTOR FIELD V on \mathbb{R}^3 is a function that assigns to each pt \vec{p} of \mathbb{R}^3 a Tangent vector $V(\vec{p})$ to \mathbb{R}^3 at \vec{p} . ; $V(\vec{p}) \in T_p \mathbb{R}^3$.

ALGEBRAIC OPERATIONS ON VECTOR FIELDS

IF V, W VFs on \mathbb{R}^3 , $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ fn.

Define VFs $V+W$ and fV on \mathbb{R}^3 by

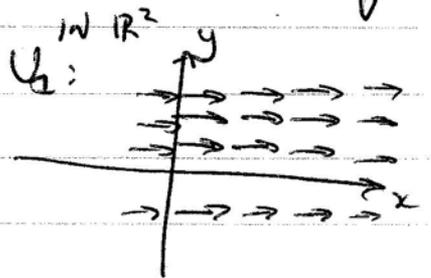
① $(V+W)(\vec{p}) = V(\vec{p}) + W(\vec{p})$
↑ Addition of vectors in vector space $T_p \mathbb{R}^3$

② $(fV)(\vec{p}) = f(\vec{p}) \cdot V(\vec{p})$
↑ Scalar multⁿ in $T_p \mathbb{R}^3$.

DEF
THE NATURAL FRAME FIELD ON \mathbb{R}^3

U_1, U_2, U_3 are the VFs on \mathbb{R}^3 defined by

$U_1(\vec{p}) = (1, 0, 0)_p$
 $U_2(\vec{p}) = (0, 1, 0)_p$
 $U_3(\vec{p}) = (0, 0, 1)_p$



At each $\vec{p} \in \mathbb{R}^3$, $\{U_1(\vec{p}), U_2(\vec{p}), U_3(\vec{p})\}$ is a BASIS for $T_p \mathbb{R}^3$

Hence

LEMMA

Let V be any VF on \mathbb{R}^3 .
Then there are functions $v_1, v_2, v_3: (\mathbb{R}^3 \rightarrow \mathbb{R})$
so that

$$V = v_1 U_1 + v_2 U_2 + v_3 U_3$$

EX VF

$$V = xy U_1 + \sin(z) U_2 + e^{yz} U_3$$

$$V(1, 2, 0) = 1 \cdot 2 \cdot U_1 + 0 U_2 + e^0 U_3 \\ = (2, 0, 1)$$

1.3DIRECTIONAL DERIVATIVES

Also See Math 251 15.3

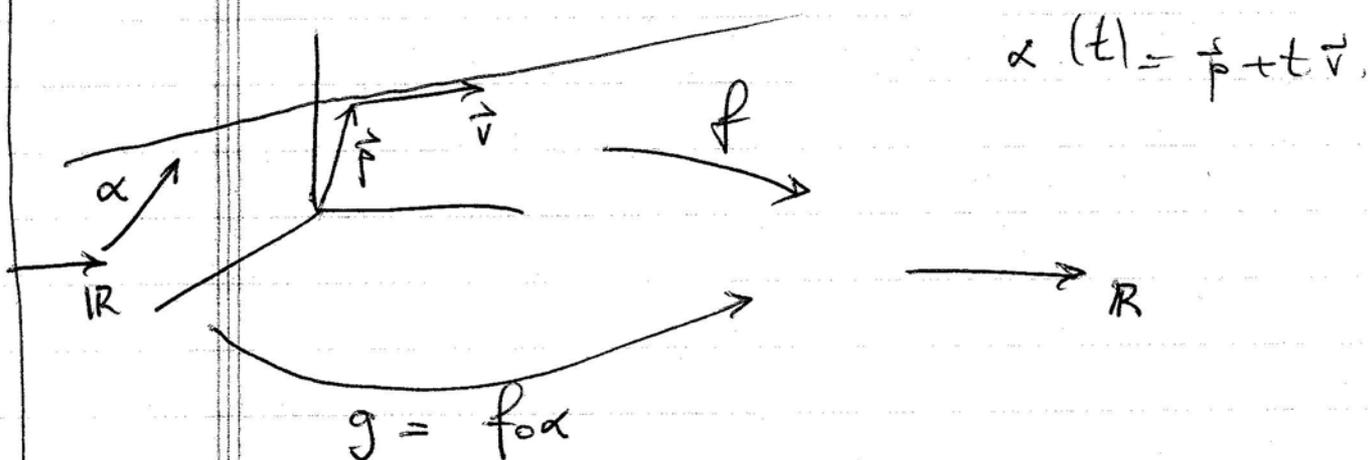
PARAMETRIZATIONS OF LINES IN \mathbb{R}^3 Let L be line through pt \vec{p} in dirn \vec{v} The ~~function~~ ^{PARAMETRIZED} CURVE (14.2) $\alpha: \mathbb{R} \rightarrow \mathbb{R}^3$ defined by $\alpha(t) = \vec{p} + t\vec{v}$ is a parametrization of L

$\alpha(t)$ = POSITION of point moving along L that is at \vec{p}
at time $t=0$ and has velocity vector \vec{v} :

$$\alpha(0) = \vec{p} \quad \alpha'(0) = \vec{v}$$

DIRECTIONAL DERIVATIVES

Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function, and $\vec{v}_p \in T_p \mathbb{R}^3$.
The directional derivative of f in direction \vec{v}_p is defined as follows:



Define $g: \mathbb{R} \rightarrow \mathbb{R}$ by $g = f \circ \alpha$, i.e.

$$g(t) = f(\alpha(t)) = f(\vec{p} + t\vec{v})$$

g is restriction of f to the line L

$$g'(0) = \text{RefC of } f \text{ at } \vec{p} \text{ in direction } \vec{v}. \quad \left(\begin{array}{l} \text{We don't require} \\ |\vec{v}| = 1 \end{array} \right)$$

DEF DIRECTIONAL DERIVATIVE of f in direction \vec{v}_p is

$$\vec{v}_p [f] = \left. \frac{d}{dt} (f(\vec{p} + t\vec{v})) \right|_{t=0} = g'(0)$$

Notation $\vec{v}_p [f]$ reflects idea that vector \vec{v}
ACTS on f to produce a real number

HOW TO CALCULATE $\vec{v}_p[f]$

LEMMA Write $\vec{v}_p = (v_1, v_2, v_3)$. Then

$$\vec{v}_p[f] = \sum_{i=1}^3 v_i \frac{\partial f}{\partial x_i}(\vec{p})$$

ie Using the Gradient ∇f of f : $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$
we have

$$\vec{v}_p[f] = \vec{v} \cdot \nabla f(\vec{p})$$

Calc ex for yourself!

PF Since $g(t) = f(\alpha(t))$ is a composition we use Chain Rule (M251, 15.6)

$$g(t) = f(\alpha(t))$$
$$g'(t) = \nabla f(\alpha(t)) \cdot \alpha'(t)$$

$$\vec{v}_p[f] = g'(0) = \nabla f(\alpha(0)) \cdot \alpha'(0)$$
$$= \nabla f(\vec{p}) \cdot \vec{v}$$

BTM Let $f, g: \mathbb{R}^3 \rightarrow \mathbb{R}$ be f^m , $\vec{v}_p, \vec{w}_p \in T_p \mathbb{R}^3$.
 $a, b \in \mathbb{R}$

① LINEARITY wrt $T_p \mathbb{R}^3$.

$$(a \vec{v}_p + b \vec{w}_p)[f] = a \vec{v}_p[f] + b \vec{w}_p[f]$$

② LINEARITY wrt f^m

$$\vec{v}_p[af + bg] = a \vec{v}_p[f] + b \vec{v}_p[g]$$

③ PRODUCT RULE

$$\vec{v}_p[fg] = \vec{v}_p[f] \cdot g(\vec{p}) + f(\vec{p}) \cdot \vec{v}_p[g]$$

OMIT
PF OMIT P

① $(a\vec{v} + b\vec{w})[f] = (a\vec{v} + b\vec{w}) \cdot \nabla f$
 $= a\vec{v} \cdot \nabla f + b\vec{w} \cdot \nabla f$ (IS LINEAR)
 $= a\vec{v}[f] + b\vec{w}[f]$

③ $\vec{v}[fg] = \sum_{i=1}^3 v_i \frac{\partial}{\partial x_i} (fg)$
 $\stackrel{\text{PR}}{\text{for } \frac{\partial}{\partial x_i}} = \sum_{i=1}^3 v_i \left[\frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \right]$
 $= \left(\sum_{i=1}^3 v_i \frac{\partial f}{\partial x_i} \right) g + f \left(\sum_{i=1}^3 v_i \frac{\partial g}{\partial x_i} \right)$
 $\stackrel{\text{LW of IP}}{=} = \vec{v}[f] g + f \vec{v}[g]$

DEF Let V be a VF on \mathbb{R}^3 $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ f^n
 Define function $V[f]: \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$(V[f])(\vec{p}) = V(\vec{p})[f] = V \cdot \nabla f(\vec{p})$$

NOTICE f and $V[f]$ are both f^n from $\mathbb{R}^3 \rightarrow \mathbb{R}$

~~GENERAL PRINCIPLE~~

CLAIM $U_1[f] = \text{RofC of } f \text{ in dirn } (1, 0, 0) = \frac{\partial f}{\partial x_1}$

PF $U_1[f] = \frac{d}{dt} \Big|_{t=0} f(\vec{p} + tU_1) = \frac{d}{dt} \Big|_{t=0} f(p_1+t, p_2, p_3)$
 $\stackrel{\text{by Defn}}{=} = \frac{\partial f}{\partial x_1}(\vec{p})$ \square

PUNCHLINE ON HOW TO COMPUTE ∇f

$$V = z U_1 - x^2 U_2 + e^y U_3$$

$$f = x \cos(y)$$

$$\nabla f \Big|_{\substack{LW \\ \text{in } \mathbb{R}^3}} = z U_1 [x \cos y] - x^2 U_2 [x \cos y] + e^y U_3 [x \cos y]$$

$$= z \frac{\partial}{\partial x} (x \cos y) - x^2 \frac{\partial}{\partial y} (x \cos y) + e^y \frac{\partial}{\partial z} (x \cos y)$$

$$\stackrel{PR}{=} z \cos y - x^2 \cdot x(-\sin y) + 0$$

$$= z \cos y + x^3 \sin y$$

$$\nabla f (1, 1, 0) = \sin 1$$

$$= \text{RfC of } f \text{ in dirn } V(1, 1, 0) = (0, -1, e)$$

at $(1, 1, 0)$

OR $\nabla_x \nabla f(p)$