MATH 423/673

1 Curves

Definition: The *velocity vector* of a curve $\alpha : I \to \mathbf{R}^3$ at time t is the tangent vector to $\mathbf{R}^3$ at $\alpha(t)$,

$$\alpha'(t) \; \in \; T_{\alpha(t)}\mathbf{R}^3$$

defined by

$$\alpha'(t) \; := \lim_{h\to 0} \frac{\alpha(t+h)-\alpha(t)}{h}.$$

Note that the algebraic operations on the right hand side are vector subtraction and scalar multiplication of a vector.

Physics meaning: $\alpha'(t)$ is the rate of change of the position of a particle, ie the velocity vector of the particle. The speed is $\|\alpha'(t)\|$.

Geometric meaning: $\alpha(t+h)-\alpha(t)$ is a secant vector which goes from the point $\alpha(t)$ to the point $\alpha(t+h)$ on the curve. If you scale it by $1/h$ the resulting vector converges to the tangent vector $\alpha'(t)$ as $h \to 0$.

Algebraically: It's not hard to show that if $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ then $\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$.

Example: "Accelerating circular motion". The curve $\alpha(t) = (\cos(t^2), \sin(t^2))$ parametrizes the circle $x^2+y^2=1$. The velocity vector at time t is $\alpha'(t) = (-2t\sin(t^2), 2t\cos(t^2))$, and so the speed is $\|\alpha'(t)\| = 2|t|$, which increases linearly with t (for $t > 0$).

Parametrization of tangent line to α at $\alpha(t)$: A parametrization of the line through $\mathbf{p}$ in direction $\mathbf{v}$ is $\ell(s) = \mathbf{p} + s\mathbf{v}$. Since the tangent line to α at $\alpha(t)$ goes through $\mathbf{p} = \alpha(t)$ in direction $\mathbf{v} = \alpha'(t)$, we conclude that this tangent line has parametrization

$$\ell(s) = \alpha(t) + s\alpha'(t).$$

Note that this is linear in s, as is expected for a line. Here

- t is the parameter on α. It tells us which tangent line we are on.
- s is the parameter on the tangent line. It tells us where we are on that tangent line.

It would be nice if ℓ passed through $\mathbf{p}$ at the same time as α did. This can be done by shifting the time s on ℓ, i.e., we can instead use the parametrization

$$\ell(s) = \alpha(t) + (s-t)\alpha'(t),$$

which is still linear in s. We think of t as being fixed in this equation. Now $\ell(t) = \alpha(t)$ and $\ell'(t) = \alpha'(t)$. So if one particle travels with parametrization α and another with ℓ, then they both pass through the point $\alpha(t)$ at same time and with same velocity. In this sense ℓ is a linearization of the motion α at $\alpha(t)$.

Example: Standard Circle:

$$\begin{aligned} (t) &= (\cos t, \sin t) \\ \alpha'(t) &= (-\sin t, \cos t) \\ \ell(s) &= (\cos t - (s-t)\sin t, \sin t + (s-t)\cos t) \end{aligned}$$

So at $t = \pi/4$ we have $(x, y) = \alpha(\pi/4) = \frac{1}{\sqrt{2}}(1, 1)$ and

$$\ell(s) = \frac{1}{\sqrt{2}}(1 - (s - \pi/4), 1 + (s - \pi/4))$$

which parametrizes the line $x + y = \sqrt{2}$. Draw a picture!

Recall: $\mathbf{v}_p[f] = \frac{d}{dt}(f \circ \beta)(0)$, where $\beta(t) = \mathbf{p} + t\mathbf{v}$ is a line. More generally we have:

Lemma 1. *Let α be any curve with $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{v}$. Then*

$$\mathbf{v}_p[f] = \frac{d}{dt}(f \circ \alpha)(0).$$

Moral: You can calculate directional derivatives using *any* curve with correct values for zero-th and first derivatives, not just using a straight line.
Proof: The is a homework problem 1.4.6. Hint: Chain rule for functions on curves.

2 Mappings between Euclidean Spaces

Here we study functions $F : \mathbf{R}^n \to \mathbf{R}^m$.

Definition 1. *Given $F : \mathbf{R}^n \to \mathbf{R}^m$, let $f_1, \cdots f_m$ be the* **coordinate functions** *of F defined by*

$$F(\mathbf{p}) = (f_1(\mathbf{p}), \cdots, f_m(\mathbf{p}))$$

where $f_j : \mathbf{R}^n \to \mathbf{R}$. If all the functions f_j are differentiable we call F a **mapping** *from $\mathbf{R}^n$ to $\mathbf{R}^m$.*

Main Idea: We study mappings using curves.

Examples:

1. $\alpha : \mathbf{R} \to \mathbf{R}^3$, **curves**

2. $f : \mathbf{R}^3 \to \mathbf{R}$, **scalar-valued functions**

3. $F : \mathbf{R}^2 \to \mathbf{R}^2$, **transformations of the plane.**

$$(u, v) \;=\; F(x, y) \;=\; (f_1(x, y), f_2(x, y))$$

To visualize F we work out where the *grid curves* $x = x_0$ and $y = y_0$ (ie horizontal and vertical lines in domain plane) get mapped to. The grid curve $x = x_0$ gets mapped to the curve $\alpha(t) = F(x_0, t)$ in the range plane. The grid curve $y = y_0$ gets mapped to the curve $\beta(t) = F(t, y_0)$ in the range plane. If F is nice (see later) these curves in the range plane can be used to define a coordinate system.

(a) **Linear transformations.** Recall that $F : \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if

$$F(\lambda\mathbf{x} + \mathbf{y}) = \lambda F(\mathbf{x}) + F(\mathbf{y}) \qquad \text{for all} \mathbf{x}, \mathbf{y} \in \mathbf{R}^n \text{ and all } \lambda \in \mathbf{R}.$$

Fact: Linear transformations map lines to lines.
Example: $F(\mathbf{x}) = A\mathbf{x}$, where A is an $m \times n$ matrix.

(b) **Linear transformations** $F : \mathbf{R}^2 \to \mathbf{R}^2$.

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

So $u = ax + by = f_1(x, y)$ and $v = cx + dy = f_2(x, y)$. Two important examples are

i. **Rotations:** Fix an angle θ. Clockwise rotation through θ is given by the linear transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The point $(1, 1)$ gets mapped to $F(1, 1) = (\cos\theta + \sin\theta, -\sin\theta + \cos\theta) = (\sqrt{2}, 0)$, if $\theta = \pi/4$. The grid curve $x = 1$ gets mapped to the line $\cos(\theta)u - \sin(\theta)v = 1$, which is the line $u - v = \sqrt{2}$ if $\theta = \pi/4$. To see this invert the matrix to solve for $\begin{pmatrix} x \\ y \end{pmatrix}$ in terms of $\begin{pmatrix} u \\ v \end{pmatrix}$ and then set $x = 1$. Try it! What line does the grid curve $y = 1$ get mapped to?

ii. **Scalings:** $F(x, y) = (ax, dy)$ (ie a diagonal matrix). This transformation maps horizontal lines to horizontal lines, vertical lines to vertical lines, squares to rectangles, circles to ellipses. Check for yourself!

(c) **Change of variables:** (*Not* linear in general.) The most useful and simplest is that from polar to rectangular coordinates.

$$(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta).$$

The grid curves $\theta = \theta_0$ get mapped to rays $y = \tan(\theta_0)x$ and the grid curves $r = r_0$ get mapped to circles $x^2 + y^2 = r_0^2$.

4. $F : \mathbf{R}^3 \to \mathbf{R}^3$, **transformations of space**.
   Example: Change of variables from spherical to rectangular coordinates

   $$(x, y, z) = F(\rho, \phi, \theta) = (\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi).$$

   Here ρ is radial distance from origin, ϕ is the "drop"angle from the north pole, and θ is the angle in the xy-plane from the x-axis (as for polar coordinates). We have $\rho > 0$, $0 < \theta < 2\pi$, $0 < \phi < \pi$. See Stewart 13.7.

5. $\mathbf{x} : \mathbf{R}^2 \to \mathbf{R}^3$, **parametrizations of surfaces in $\mathbf{R}^3$**. See Stewart 13.7, 17.6. We will study these more later on.
   Example: Parametrization of unit sphere using longitude and latitude.

   $$(x, y, z) = \mathbf{x}(\phi, \theta) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi).$$

Goal: Given $F : \mathbf{R}^n \to \mathbf{R}^m$ we want to define the **tangent map** of F at $\mathbf{p}$:

$$F_* : T_p\mathbf{R}^n \to T_{F(p)}\mathbf{R}^m$$

and show that

1. F_* is a linear transformation (and hence a matrix transformation)
2. The matrix of F_* with respect to the standard bases for $T_p\mathbf{R}^n$ and $T_{F(p)}\mathbf{R}^m$ is the matrix of partial derivatives of F.

Definition 2. *If $\alpha : I \to \mathbf{R}^n$ is a curve, the* **image of α under a mapping** $F : \mathbf{R}^n \to \mathbf{R}^m$ *is the curve $\beta : I \to \mathbf{R}^m$ defined by $\beta = F \circ \alpha$, i.e., $\beta(t) = F(\alpha(t))$.*

Example: If $\mathbf{x}$ is the parametrization of unit sphere given above and $(\phi, \theta) = \alpha(t) = (t, \pi/4)$ is the grid curve $\theta = \pi/4$ in the domain space $\mathbf{R}^2$ of $\mathbf{x}$, i.e. in (ϕ, θ)-space, then the image of α under $\mathbf{x}$ is the line of longitude

$$\beta(t) = \mathbf{x}(\alpha(t)) = \mathbf{x}(t, \pi/4) = (\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\cos t, \sin t).$$

Definition 3. *The* **tangent map**, $F_* : T_p\mathbf{R}^n \to T_{F(p)}\mathbf{R}^m$, *of a mapping* $F : \mathbf{R}^n \to \mathbf{R}^m$ *is defined as follows. Let* $\mathbf{v} \in T_p\mathbf{R}^n$. *Define* $\alpha(t) = \mathbf{p} + t\mathbf{v}$, *which is a curve (line!) in* $\mathbf{R}^n$ *with* $\alpha(0) = \mathbf{p}$ *and* $\alpha'(0) = \mathbf{v}$. *Set* $\beta = F \circ \alpha$ *to be the image of* α *under* F. *So* $\beta(t) = F(\alpha(t)) = F(\mathbf{p} + t\mathbf{v})$. *Then we define*

$$F_*(\mathbf{v}) \;=\; \beta'(0) \qquad \in T_{F(p)}\mathbf{R}^m.$$

Proposition 1. *"The tangent map of* F *is given by the directional derivatives of the coordinate functions of* F*."Specifically: Let* $F = (f_1, \cdots, f_m) : \mathbf{R}^n \to \mathbf{R}^m$ *and* $\mathbf{v}_p \in T_p\mathbf{R}^n$. *Then*

$$F_*(\mathbf{v}) \;=\; (\mathbf{v}[f_1], \cdots, \mathbf{v}[f_m]) \qquad \text{at } F(\mathbf{p}).$$

Proof: See proof of Proposition 1.7.5 on page 37-38 of O'Neill.

Example: Let $(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta)$ and $\mathbf{v} = (1, 2)$ at $\mathbf{p} = (1, \pi/4)$. Then

$$\begin{aligned} F_*(\mathbf{v}) &= (\mathbf{v}[r\cos\theta], \mathbf{v}[r\sin\theta]) \\ &= (\mathbf{v}\cdot\nabla f_1, \mathbf{v}\cdot\nabla f_2) \\ &= (\mathbf{v}\cdot(\cos\theta, -r\sin\theta), \mathbf{v}\cdot(\sin\theta, r\cos\theta)) \\ &= \left((1,2)\cdot(\frac{1}{\sqrt{2}}, -1\frac{1}{\sqrt{2}}), (1,2)\cdot(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\right) \;=\; (\frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}}). \end{aligned}$$

Corollary 1. $F_* : T_p\mathbf{R}^n \to T_{F(p)}\mathbf{R}^m$ *is a linear transformation.*

Proof: See proof of Corollary 1.7.6 on page 38 of O'Neill.

Meaning: (See UMBC's yet-to-exist Advanced Calculus course, or Math 302 if you are lucky.) The tangent map F_* at $\mathbf{p}$ is the linear transformation that "best approximates" F near $\mathbf{p}$. We also say that the tangent map is the "linearization "of F at $\mathbf{p}$.

Example: If $F : \mathbf{R} \to \mathbf{R}$ then $F : T_p\mathbf{R} \to T_{F(p)}\mathbf{R}$ is a linear transformation between one-dimensional vector spaces. So it is given by a 1×1 matrix, ie, it is of the form $F_*(\mathbf{v}) = a\mathbf{v}$ for some real number a. You can check using the Proposition above that

$$F_*(\mathbf{v}) \;=\; F'(p)\mathbf{v}$$

is scalar multiplication by $F'(p)$. [Hint: Choose the vector $\mathbf{v}$ to be $\mathbf{v} = 1$ and use linearity.]

Corollary 2. *Let* $U_1, \cdots U_n$ *be the natural frame field on* $\mathbf{R}^n$ *and* $\overline{U}_1, \cdots \overline{U}_m$ *be the natural frame field on* $\mathbf{R}^m$. *If* $F = (f_1, \cdots, f_m) : \mathbf{R}^n \to \mathbf{R}^m$, *then*

$$F_*(U_j(\mathbf{p})) \;=\; \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(\mathbf{p})\overline{U}_i(F(\mathbf{p})), \qquad j = 1, \cdots, n.$$

Meaning: Let DF be the $m \times n$ matrix with

$$(DF)_{ij} \;=\; \frac{\partial f_i}{\partial x_j}$$

So the i-th row of DF is the vector given by the gradient of f_i:

$$DF \;=\; \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_m \end{pmatrix}.$$

DF is the matrix of partial derivatives of F, otherwise known as the **Jacobian matrix** of F.

Recall: If W is a vector space with basis $\mathcal{C} = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$, then the coordinate vector of w is the vector $[w]_{\mathcal{C}} \in \mathbf{R}^m$ given by

$$[w]_{\mathcal{C}} = (\lambda_1, \cdots, \lambda_m)$$

where

$$w = \sum_{i=1}^{m} \lambda_i w_i.$$

Also, if $T : V \to W$ is a linear transformation from an n-dimensional vector space V with basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ to an m-dimensional vector space W with basis $\mathcal{C} = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$, then the matrix of T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ is given by

$$[T]_{\mathcal{B}C} \;=\; \left([T(\mathbf{v}_1)]_{\mathcal{C}}, \cdots [T(\mathbf{v}_n)]_{\mathcal{C}}\right),$$

i.e., the j-th column of the matrix of T is the coordinate vector in the basis for W of the image under T of the basis vector $\mathbf{v}_j$ of V.

In this language, the Jacobian matrix, DF, is the matrix of the linear transformation F_* with respect to the standard bases $U_1(\mathbf{p}), \cdots U_n(\mathbf{p})$ for $T_p\mathbf{R}^n$ and $\overline{U}_1(F(\mathbf{p})), \cdots \overline{U}_m(F(\mathbf{p}))$ for $T_{F(p)}\mathbf{R}^m$. Check:

$$[F_*] = [F_*(U_1), \cdots F_*(U_n)] = \left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}\right)$$

where $\frac{\partial F}{\partial x_j}$ is the column vector whose i-th row is $\frac{\partial f_i}{\partial x_j}$.

Example: $(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta)$.

$$DF = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{pmatrix} \;=\; \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Proof of Corollary 2:

$$F_*(U_j(\mathbf{p})) \;=\; (U_j[f_1], \cdots, U_j[f_m]) \;=\; \sum_{i=1}^{m} U_j[f_i]\overline{U}_i(F(\mathbf{p}))$$

and

$$U_j[f_i] \;=\; U_j \cdot \nabla f_i = (0, ..., 1, ..., 0) \cdot \left(\frac{\partial f_i}{\partial x_1}, \cdots, \frac{\partial f_i}{\partial x_n}\right) \;=\; \frac{\partial f_i}{\partial x_j}.$$