MATH 423/673

1 Inverse Function Theorem and Regular Mappings

Definition 1. *1. A function $f : X \to Y$ is* **invertible** *if there is a function $g : Y \to X$ so that $f \circ g = \mathrm{Id}_Y$ (i.e. $f(g(y) = y$ for all $y \in Y$) and $g \circ f = \mathrm{Id}_X$.*

2. If f is invertible, then the inverse g is unique and is denoted by f^{-1}.

3. When f is a mapping (ie is differentiable) we augment the definition of the invertibility of f to include the condition that f^{-1} is also differentiable.

4. A mapping that has a (differentiable) inverse mapping is called a **diffeomorphism**.

Fact: f is invertible if and only if f is 1-1 and onto.

Question: When does a mapping $F : \mathbf{R}^n \to \mathbf{R}^m$ have an inverse?

The answer is given by the **Inverse Function Theorem**. Let's start with two special cases.

Special case I: Linear Transformations

1. If a linear transformation $F : \mathbf{R}^n \to \mathbf{R}^m$ is invertible then the dimensions of the domain and range vector spaces must be equal: $n = m$.
2. An invertible linear transformation is sometimes called a **linear isomorphism**.
3. There are (at least) 24 equivalent criteria that each gaurantee that a linear transformation $F : \mathbf{R}^n \to \mathbf{R}^n$ is invertible, including:
   (a) F is 1-1
   (b) F is onto
   (c) $\mathrm{Nullspace}(F) = \{\mathbf{0}\}$
   (d) $\mathrm{Rank}(F) = n$.
   (e) The matrix A of F in any basis for $\mathbf{R}^n$ is invertible
   (f) $\det(A) \neq 0$
4. It turns out that the condition $n = m$ is necessary for an arbitrary mapping F to be invertible.

Special case II: Differentiable functions $F : \mathbf{R} \to \mathbf{R}$.

1. In order for F to be invertible we need that either $F'(x) > 0$ for all x (F strictly increasing) or $F'(x) < 0$ for all x (F strictly decreasing).

2. Equivalently: If F' is continuous, we just need that $F'(x) \neq 0$ for all x, i.e., that the **tangent map** $F_* : T_x\mathbf{R} \to T_{F(x)}\mathbf{R}$ **is an invertible linear map** (see Special Case I!).

3. This suggests that the invertibility of a general mapping F is determined by the invertibility of all of its linearizations (ie of its tangent mappings).

4. The condition that F be strictly monotonic is too restrictive. Instead it is better to ask when a mapping is **locally invertible**.

Definition 2. *A mapping $F : \mathbf{R}^n \to \mathbf{R}^n$ is* **locally invertible at $\mathbf{p} \in \mathbf{R}^n$** *if there is an open set $\mathcal{U}$ containing p and an open set $\mathcal{V}$ containing $F(p)$ so that the restriction of F to $\mathcal{U}$, $F\big|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$, is an invertible mapping.*

Example: $F(x) = x^2$ is not invertible on $\mathbf{R}$ but it is locally invertible at all points x_0 except for $x_0 = 0$.

A picture suggests that if $F'(x_0) \neq 0$ then F is indeed locally invertible at x_0, ie, if the tangent mapping $F_* : T_{x_0}\mathbf{R} \to T_{F(x_0)}\mathbf{R}$ at x_0 is an invertible linear transformation then $F : \mathbf{R} \to \mathbf{R}$ is locally invertible at x_0.

More generally we have

Theorem 1 (The Inverse Function Theorem)**.** *Let $F : \mathbf{R}^n \to \mathbf{R}^n$. If F_* is invertible at $\mathbf{p}$ then F is locally invertible at $\mathbf{p}$. Furthermore, in this case the matrix of partial derivatives of the inverse mapping is the inverse of the matrix of partial derivatives of the original mapping, ie,*

$$D(F^{-1}) \;=\; (DF)^{-1}.$$

Notes:

1. The point is that we can use linear algebra (ie any of the 24 criteria alluded to above) to easily check whether or not F_* is invertible at $\mathbf{p}$.

2. The proof of the Inverse Function Theorem is sometimes done in Math 302. It uses the Contraction Mapping Theorem.

3. The Inverse Function Theorem says: If F_* is a linear isomorphism at $\mathbf{p}$ then F is a local diffeomorphism at $\mathbf{p}$.

If $n \neq m$ the Inverse Function Theorem does not apply. However it is still useful to know if F_* is 1-1.

Definition 3. *A mapping $F : \mathbf{R}^n \to \mathbf{R}^m$ is* **regular** *if for all $\mathbf{p} \in \mathbf{R}^n$, $F_* : T_{\mathbf{p}}\mathbf{R}^n \to T_{F(\mathbf{p})}\mathbf{R}^m$ is 1-1.*

Example: If a curve $\alpha : I \to \mathbf{R}^3$ is regular then α has a tangent line at each point $\alpha(t) \in \mathbf{R}^3$. In fact this tangent line is the one-dimensional vector subspace $\alpha_*(T_t\mathbf{R}) \subset T_{\alpha(t)}\mathbf{R}^3$. i

Note: The image of the curve $\alpha(t) = (t^3, t^2)$ has a cusp at the origin. α fails to be regular at $t = 0$ and has no tangent line at $(0, 0)$.