MATH 423/673

1 Curves

Let $\alpha : I \to \mathbf{R}^3$ be a curve.

- Position: $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$
- Velocity: $\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$
- Speed: $\|\alpha'(t)\|$
- Acceleration: $\alpha''(t) = (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t))$
- Length of α: $L = \int_I \|\alpha'(t)\|\, dt$ from Distance = Speed x Time formula
- Note that $\alpha'(t)$ and $\alpha''(t)$ are **vector fields along** α.

Definition 1. *A* **vector field***, V,* **along a curve** *α assigns to each $t \in I$ a tangent vector, $V(t) \in T_{\alpha(t)}\mathbf{R}^3$, to $\mathbf{R}^3$ at $\alpha(t)$. (Read pages 52-54 of O'Neill for more background.)*

Example: (Helix)

- Position: $\alpha(t) = (\cos t, \sin t, t)$
- Velocity: $\alpha'(t) = (-\sin t, \cos t, 1)$
- Speed: $\|\alpha'(t)\| = \sqrt{2}$
- Acceleration: $\alpha''(t) = (-\cos t, -\sin t, 0)$
- The helix is a curve on the cylinder $x^2+y^2 = 1$. As it goes once around the $+z$-axis (according to the right hand rule) it ascends 2π in the z-direction. The acceleration vector always goes from a point on the helix to the point on the z-axis that is at the same height. Sketch the cylinder, helix, and vectors $\alpha(\pi/4)$, $\alpha'(\pi/4)$, and $\alpha''(\pi/4)$ for yourself. Note that $\alpha(\pi/4)$ starts at origin but $\alpha'(\pi/4)$ and $\alpha''(\pi/4)$ start at $\alpha(\pi/4)$.

Problem: Given a curve α from $\mathbf{p}$ to $\mathbf{q}$ there are an *infinite* number of curves β with the same image as α. For example β could have a different speed function than α.

Solution: There is a *canonical* curve β that has the same image as α. It is the curve from $\mathbf{p}$ to $\mathbf{q}$ that has constant speed 1: $\|\beta'(s)\| = 1$ for all s.

Question: Given any curve α how do we find this unit speed curve β?

Reparametrizations of α (See O'Neill, 1.4 pp 19-20).

Definition 2. *Suppose* $\alpha : (a,b) \to \mathbf{R}^3$ *and* $g : (c,d) \to (a,b)$ *is any function. Then* $\beta = \alpha \circ g : (c,d) \to \mathbf{R}^3$ *is called a* **reparametrization** *of* α*.*

- If $\alpha = \alpha(t)$ and $t = g(s)$ then $\beta(s) = \alpha(g(s))$
- t is the parameter of the original curve
- s is the parameter of the reparametrized curve.
- If $g(c) = a$, $g(d) = b$, and g is an increasing function, then β traverses the same route as α but at a different speed.
- In fact, by the Chain Rule, we have

$$\beta'(s) \; = \; \alpha'(g(s))g'(s)$$

  so if $g' > 0$ then the velocity vectors β' and α' are both tangent to the curve and point in the same direction (so no back-tracking can occur).

- So

$$\|\beta'(s)\| \; = \; \|\alpha'(g(s))\| \, |g'(s)| \tag{1}$$

Theorem 1. *If* α *is a regular curve (i.e.,* $\alpha'(t) \neq 0$ *for all* t*), then there is a unit speed reparametrization,* β*, of* α*. In this case, the parameter,* s*, for* β *is arclength from the starting point.*

Note: This makes sense: If the time parameter for a curve is its arclength then the distance traveled equals time taken and so speed must be one.

Proof: We first define the **arclength function** along α by

$$s \; = \; L(t) \; = \; \int_a^t \|\alpha'(u)\| \, du$$

to be the length of α from $\alpha(a)$ to $\alpha(t)$.
By the Fundamental Theorem of Calculus:

$$L'(t) \; = \; \|\alpha'(t)\| \; > \; 0 \tag{2}$$

as α is regular.
Therefore L is a strictly increasing function. So it has an inverse function

$$g : (0, L(b)) \to (a,b).$$

Properties:

1. Since $(L \circ g)(s) = s$ we have $(L \circ g)'(s) = 1$

2. So by Inverse Function Theorem,

$$g'(s) \;=\; \frac{1}{L'(g(s))} \;>\; 0 \qquad \text{by (2).}$$

Since s is areclength we define

$$\beta(s) \;=\; \alpha(g(s)).$$

Let's check that β really does have unit speed. Well

$$\begin{aligned} \|\beta'(s)\| &= \|\beta'(s)\|\,|g'(s)| \qquad \text{by (1)} \\ &= L'(g(s))\,g'(s) \qquad \text{by (2) and second property above} \\ &= (L \circ g)'(s) \qquad \text{by Chain Rule} \\ &= 1 \qquad \text{by first property above.} \qquad \square \end{aligned}$$

Caution: Usually it is impossible to find an explicit formula for the arclength reparametrization of α, as it is hard/impossible to calculate an explicit formula for the arclength function (it involves the integral of a square root of a sum of squares) and even if you could it is often then hard/impossible to find an explicit formula for the inverse of this arclength function.

Cooked-up Example: (Where everything works!)

$$\alpha(t) \;=\; (t\cos t, t\sin t, \frac{1}{\sqrt{2}}t^2) \qquad 0 < t < 1.$$

This curve lies on the paraboloid $x^2 + y^2 = \sqrt{2}z$ and spirals up it from the origin.

$$\|\alpha'(t)\| \;=\; \sqrt{(t+1)^2} = t + 1 \qquad \text{Cooked up so no square roots!!}$$

So

$$s \;=\; L(t) = \int_0^t u + 1\, du \;=\; \frac{1}{2}t^2 + t$$

Find inverse, g of L, by solving for $t = g(s)$ (Cooked up as can always find roots and hence inverse of a quadratic):

$$\begin{aligned} \frac{1}{2}t^2 + t &= s \\ (t+1)^2 &= 2s + 1 \\ t &= \sqrt{2s+1} - 1 \;=\; g(s) \end{aligned}$$

So

$$\beta(s) \;=\; \alpha(\sqrt{2s+1}-1) \;=\; ((\sqrt{2s+1}-1)\cos(\sqrt{2s+1}-1), (\sqrt{2s+1}-1)\sin(\sqrt{2s+1}-1), \frac{1}{\sqrt{2}}(\sqrt{2s+1}-1)^2)$$

You can check (!) that $\|\beta'(s)\| = 1$.