7.7 NILPOTENT MATRICES + JORDAN STRUCTURE

DEF An $n \times n$ L is NILPOTENT with INDEX k if

$$L^k = 0 \quad \text{but} \quad L^{k-1} \neq 0.$$

LEMMA II

If L is nilpotent index k

Then for all $j \leq k$

$$R(L^{j+1}) \subsetneq R(L^j)$$

PF

Clearly $R(L^{j+1}) \subseteq R(L^j)$

Suppose $R(L^{j+1}) = R(L^j)$ for some $j < k$ $\circledast$

Let $\vec{x}$: $0 \neq L^{k-1}\vec{x}$

Then $0 \neq L^{k-1}\vec{x} = L^{k-1-j}(L^j \vec{x})$ $\quad j < k \Rightarrow k-1-j \geq 0$

$= L^{k-1-j}(L^{j+1}\vec{y})$ by $\circledast$

$= L^k \vec{y}$

So $L^k \neq 0$ $\quad \lightning$ $\quad \square$

LEMMA II (UBV)

(1) $\sigma(L) = \{0\}$.

(2) So if $L \neq 0$ $(N(L) \neq \mathbb{R}^n)$ Then L is not diagonalizable

SCHUR LEMMA (OMIT PF)

For any $n \times n$ A, A is similar to an upper Δ^r.

NOTE

Since diagonal entries of upper Δ^r matrix are its evalues (UBV), we must have

L similar to $\begin{pmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{pmatrix}$

IN FACT we can do better

$\exists P$:

$$P^{-1}LP = \begin{pmatrix} 0 & * & & 0 \\ & \ddots & \ddots & \\ & & \ddots & * \\ 0 & & & 0 \end{pmatrix}$$

Upper Super Diagonal as non zero.

BASIC EXS

① $L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $n=3$.

$L^3 = 0$ but $L^2 \neq 0$. Index is $k=3$.

$L\vec{e}_1 = \vec{0}$
$L\vec{e}_2 = \vec{e}_1$
$L\vec{e}_3 = \vec{e}_2$

OR $\vec{e}_1 = L^2\vec{e}_3$, $\vec{e}_2 = L^1\vec{e}_3$, $\vec{e}_3 = L^0\vec{e}_3$

$N(L) = \{\vec{e}_1\}$

② Let L be any 3×3 Nilpotent with Index $k=3$.

Then $\exists \vec{v}: \vec{x} = L^2\vec{v} \neq 0$ but $L\vec{x} = 0$.

(This $\vec{x}$ is a SPECIAL vector in $N(L)$)

CLAIM
$B = \{L^2\vec{v}, L\vec{v}, \vec{v}\}$ is a basis for L
(B is a JORDAN CHAIN for L

and $[L]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = N$

i.e. $L = PNP^{-1}$ where $P = [L^2\vec{v}, L\vec{v}, \vec{v}]$

(3) $L = \left(\begin{array}{ccc|cccc} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & 0 & \\ 0 & 0 & 0 & & & & \\ \hline & & & 0 & 0 & 0 & 0 \\ & 0 & & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 & 0 \end{array}\right)$

$n = 7$

NOTE

$\text{Index}(L) = 4$

$N(L) = \{\vec{e}_1, \vec{e}_4\}$

$\vec{e}_1 = L^2\vec{e}_3, \quad \vec{e}_2 = L\vec{e}_3, \quad \vec{e}_3 = L^0\vec{e}_3$

$\vec{e}_4 = L^3\vec{e}_7 \quad \vec{e}_5 = L^2\vec{e}_7 \quad \vec{e}_6 = L\vec{e}_7 \quad \vec{e}_7 = L^0\vec{e}_7$

So Here we have $L^2\vec{e}_3 \in N(L)$

with Jordan Chain $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

and $L^3\vec{e}_7 \in N(L)$ with Jordan Chain

$\{\vec{e}_4, \vec{e}_5, \vec{e}_6, \vec{e}_7\}$

The union of these 2 Jordan Chains is a basis (std basis) for $\mathbb{R}^7$

CLAIM $L = \begin{pmatrix} 0 & * & & \\ & & * & \\ & & & 0 \end{pmatrix} \Rightarrow L^2 = \begin{pmatrix} 0 & 0 & * & \\ & & & * \\ & & & 0 \\ & & & & 0 \end{pmatrix}$ ETC

NOTICE

$$0 \subsetneq \underset{\text{Span}(\vec{e}_4)}{R(L^3)} \subsetneq \underset{\text{Span}\{\vec{e}_1, \vec{e}_4, \vec{e}_5\}}{R(L^2)} \subsetneq \underset{\text{Span}\{\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5, \vec{e}_6\}}{R(L)} \subseteq \mathbb{R}^7$$

and

$$0 \subsetneq \underset{\text{Span}(\vec{e}_4)}{R(L^3) \cap N(L)} \subsetneq \underset{\text{Span}\{\vec{e}_1, \vec{e}_4\}}{R(L^2) \cap N(L)} = R(L) \cap N(L) = N(L^2)$$

So to build basis we ① started with basis for $R(L^3) \cap N(L)$, namely $\vec{e}_4$.

Wrote that as $\vec{e}_4 = L^3 \vec{e}_7$ and built Jordan Chain ~~from $\vec{e}_7$~~.

② We then extended basis for $R(L^3) \cap N(L)$ to one for $R(L^2) \cap N(L)$ ~~to~~ get $\vec{e}_1$

Wrote $\vec{e}_1 = L^2 \vec{e}_3$ and built Jordan Chain

~~from~~ $\vec{e}_3$.

GENERAL CONSTRUCTION

Let L be nilpotent index k.

Define $m_j = R(L^j) \cap N(L)$ $\quad j = 0, 1, \ldots, k$.

Then

$$\{\vec{0}\} = m_k \subseteq m_{k-1} \subseteq \cdots \subseteq m_1 \subseteq m_0 = N(L).$$

Let $\mathcal{S}_{k-1}$ be basis for m_{k-1}

Extend: $\mathcal{S}_{k-1} \cup \mathcal{S}_{k-2}$ basis for m_{k-2}. ETC

$$B = \mathcal{S}_{k-1} \cup \cdots \cup \mathcal{S}_0 = \{\vec{b}_1, \ldots, \vec{b}_t\}$$ basis for $N(L)$

NOTE Some $\mathcal{S}_j$ may be empty.

Re $\vec{b}_j$'s are starting vectors for our Jordan Chains

Let $\vec{b} \in \mathcal{S}_j$ be one of these.

Then $\vec{b} \in R(L^j) \cap N(L)$

So $\exists \vec{x} \in \mathbb{R}^n : \quad L^j \vec{x} = \vec{b}$

Let

$$J_{\vec{b}} = \{ L^j \vec{x}, \; L^{j-1} \vec{x}, \ldots, L\vec{x}, \; \vec{x} \}$$

be Jordan chain built from $\vec{b}$.

NOTE $\quad \# J_{\vec{b}} = j+1 \quad$ where $\quad \vec{b} \in \mathcal{S}_j$

Let

J_i be matrix whose cols are Jordan Chain built from $\vec{b}_i$, $i = 1 - t$.

Form

$$P = [J_1 \; \cdots \; J_t].$$

LEMMA III

$$LP = PN$$

where

$$N = \begin{pmatrix} N_1 & & \\ & N_2 & \\ & & \ddots \\ & & & N_t \end{pmatrix}$$

with each N_i of size given by $\#(J_{b_i})$.

and of form

$$N_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

PF

Write $J_i = [L^i x \mid L^{i-1} x \mid \dots \mid Lx \mid x]$

where $L^i x = b_i \in N(L)$

So

$$LJ_i = [0 \mid L^i x \mid L^{i-1} x \mid \dots L^2 x \mid Lx]$$

$$= J_i \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = J_i N_i$$

So

$$L[J_1 \dots J_t] = [J_1 \dots J_t] \begin{bmatrix} N_1 & & \\ & \ddots & \\ & & N_t \end{bmatrix}$$

$$LP = PN$$

$\square$

HWK Find all Jordan Forms for 4×4 Nilpotent Matrices (J_4)

LEMMA IV

$B = \bigcup_{i=1}^{t} J_{b_i}$ is a basis for $\mathbb{R}^n$.

So P is $n \times n$, INV.

PF OMIT

Hence

THM Let N be nilpotent order k.
Then $\exists P$ INV!

$$P^{-1} L P = N = \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_t \end{pmatrix}$$

where $N_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$

and $P = [J_1 \cdots J_t]$ as above

At least one of N_i has size $k \times k$

DEF N is called the Jordan Form for L.
N_i are Jordan Blocks

THM Any 2 Jordan Forms of L have same # Blocks ~~with~~ ~~same~~ and sizes of blocks match up.