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TRANSIENT STIMULATED RAMAN SCATTERING*

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Abstract. The system: $u_\xi = -zv$, $v_\xi = \bar{z}u$, $z_\tau = u\bar{v} - \gamma z$ with $z \to 0$ at $\tau \to -\infty$ and initial data for $\mathbf{u} = (u, v)$ at $\xi = 0$ are considered. Well posedness results are obtained for this and also for a version discretized in τ. Stability is considered as $\xi \to \infty$.

Key words. Raman scattering, well posed, partial differential equations, system, stability

AMS(MOS) subject classifications. 35B40, 35Q60, 78A60

1. Introduction. The Raman effect has played a conspicuous role in physics since its discovery in the 1920s [14], [11]. Specifically, the system of partial differential equations

$$\text{(1.1)} \qquad \begin{cases} \text{(i)} & \partial u/\partial\xi = -zv, \\ \text{(ii)} & \partial v/\partial\xi = \bar{z}u, \\ \text{(iii)} & \partial z/\partial\tau = u\bar{v} - \gamma z, \end{cases}$$

was first derived to model the interaction of two laser beams with gases when the frequency difference between the beams corresponds to a resonance of the gas molecules [16], [1]. Here, u and v are unknown $\mathbb{C}$-valued functions on $\mathbb{R}_+ \times \mathbb{R}$ (i.e., functions of (ξ, τ) with $0 < \xi < \infty$; $-\infty < \tau < \infty$) which represent the two laser beams, usually referred to as the pump beam and the Stokes beam, respectively. Then the function $-iz$ corresponds to the off-diagonal density matrix element which describes the quantum mechanical state of the gas. The real parameter $\gamma \geq 0$ represents a de-excitation rate due to molecular collisions.

In recent years, these equations have been the focus of intense activity, both experimental and theoretical; some references to the relevant physical literature are provided in our bibliography. It has long been known that (1.1) has a Lax pair when $\gamma = 0$ and so has soliton solutions [2]. On the other hand, we note that it is physically reasonable to require that

$$\text{(1.2)} \qquad z(\tau) \to 0 \quad \text{as } \tau \to -\infty$$

so that z should be "independent of the infinite past"; yet it was later shown that this physical boundary condition leads to special difficulties which require modification of the standard inverse scattering approach [8], [15], [9]. These modifications seriously complicate the theory, leading to results which are difficult to interpret [10]. We note that soliton-like pulses have been observed in experiments with laser beams whose durations are long compared to the collisional de-excitation time [3], [17] but,

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surprisingly, soliton-like pulses are not observed in experiments with laser beams whose durations are short compared to the molecular de-excitation time [4], [5] — even though setting $\gamma = 0$ is presumably "more legitimate" for the latter case. Indeed, numerical experiments indicate that both u and z tend toward zero almost everywhere as $\xi \to \infty$ for a fairly broad set of initial data [13], [7]; compare §5. Following the somewhat less formal argument of [12], we show here that $z \to 0$ as $\xi \to \infty$. The more detailed asymptotic behavior of u and v as $\xi \to \infty$ remains an open question and, as will become evident in the course of this paper, a somewhat delicate one.

Clearly, there is a need for careful mathematical work. Remarkably, despite the importance of (1.1) in the physics literature, no one until now has even shown that these equations are well posed! The goal of this paper is to place the study of (1.1) on a firm mathematical foundation by demonstrating well posedness, obtaining a number of other simple results relating to the asymptotic behavior of these equations as $\xi \to \infty$, and outlining the remaining difficulties and some open problems. The key insights[1] will be that solutions satisfy the identities

$$|\mathbf{u}(\xi,\tau)|^2 = |\mathbf{u}_0|^2 \quad \text{a.e. } \tau \in \mathbb{R} \quad \text{for all } \xi \geq 0 \tag{1.4}$$

(where $|\mathbf{u}_0|^2 = |u_0(\tau)|^2 + |v_0(\tau)|^2$; see (2.1)) and, also pointwise in τ,

$$\frac{d}{d\xi}\left(\int_{-\infty}^{\tau} |e^{-\gamma(\tau-\tilde{\tau})}u|^2\right) = -\frac{d}{d\xi}\left(\int_{-\infty}^{\tau} |e^{-\gamma(\tau-\tilde{\tau})}v|^2\right) = -|z(\tau)|^2. \tag{1.5}$$

2. Formulation.

2. Formulation. We will use a subscript $_\xi$ (or simply $'$) for (partial) differentiation with respect to ξ and subscript $_\tau$ for differentiation with respect to τ, etc. We will consistently use the notation

$$\mathbf{u} := (u, v), \quad X := \begin{pmatrix} 0 & -z \\ \overline{z} & 0 \end{pmatrix}, \quad \mathbf{u}_0 := (u_0, v_0),$$

$$K^2(\tau) := |\mathbf{u}_0(\tau)|^2 := [|u_0(\tau)|^2 + |v_0(\tau)|^2], \tag{2.1}$$

$$\sigma = \sigma(\tau) := \int_{-\infty}^{\tau} K^2(\tilde{\tau})\, d\tilde{\tau} \quad \text{so } K^2 d\tau =: d\sigma.$$

We will assume the initial data $\mathbf{u}(0) = \mathbf{u}_0$ is to be in $\mathcal{H}$ so

$$\kappa^2 := \|K\|^2 := \int_{-\infty}^{\infty} K^2\, d\tau < \infty.$$

We can solve (1.1.iii) as an ordinary differential equation in τ, temporarily ignoring the ξ dependence, to obtain

$$z(\tau) = e^{-\gamma(\tau-\tau_*)}z(\tau_*) + \int_{\tau_*}^{\tau} e^{-\gamma(\tau-\tilde{\tau})}u(\tilde{\tau})\overline{v(\tilde{\tau})}\, d\tilde{\tau}$$

[1] We note also that the system (1.1) is invariant under the action of the group $\mathcal{G} = \{g_\vartheta : \vartheta : \mathbb{R} \to \mathbb{R}\}$ of transformations

$$g = g_\vartheta : (u, v) \mapsto (e^{i\vartheta}u, e^{i\vartheta}v) \tag{1.3}$$

for "arbitrary" real $\vartheta = \vartheta(\tau)$, independent of ξ. So far, however, we have not been able to exploit this insight effectively.

for arbitrary real τ_*, τ. Imposing (1.2), the first term on the right can be omitted "at $-\infty$" so the differential equation (1.1.iii) can replaced by

$$z(\tau) := \int_{-\infty}^{\tau} e^{-\gamma(\tau-\tilde{\tau})} u(\tilde{\tau})\overline{v(\tilde{\tau})}\, d\tilde{\tau} \tag{2.2}$$

as a *definition*. We note that a principal point of difference between our present treatment and most previous work is precisely this imposition of the boundary condition (1.2); compare Remark 4.4 below.

For (2.2) to be meaningful, we need $u\bar{v}$ to be integrable and we will therefore seek solutions in the L^2 space[2]

$$\mathcal{H} := \{\mathbf{u} = (u, v) : \mathbb{R} \to \mathbb{C}^2 : \|\mathbf{u}\|^2 := \int_{-\infty}^{\infty} |\mathbf{u}(\tau)|^2\, d\tau < \infty\}.$$

Along with $\mathcal{H}$, we introduce the spaces

$$\begin{aligned} \mathcal{H}_K &:= \{\mathbf{u} = (u, v) \in \mathcal{H} : |\mathbf{u}(\tau)| = K(\tau) \text{ a.e. } \tau \in \mathbb{R}\}, \\ \mathcal{Z} &:= \{z \in C((-\infty, \infty] \to \mathbb{C}) : z(-\infty) = 0\}, \\ \mathcal{X} &:= \left\{ X = \begin{pmatrix} 0 & -z \\ \bar{z} & 0 \end{pmatrix} : z \in \mathcal{Z} \right\}. \end{aligned}$$

Note that $\sup\{|z(\cdot)|\}$ just gives the norm of $X(\cdot) \in \mathcal{X}$ as an operator on $\mathcal{H}$ (or on any $\mathcal{H}_K \subset \mathcal{H}$), acting by pointwise multiplication. We will also introduce the linear space $\boldsymbol{\mathcal{U}}$ of functions $\mathbf{u}(\cdot) \in C(\mathbb{R}_+ \to \mathcal{H})$ for which the exponentially weighted norm

$$\|\mathbf{u}\|_\kappa := \sup_{\xi \geq 0} \{e^{-2\kappa^2 \xi} \|\mathbf{u}(\xi, \cdot)\|_{\mathcal{H}}\} \tag{2.3}$$

is finite for some κ. Convergence in $\boldsymbol{\mathcal{U}}$ is given by convergence in $\|\cdot\|_\kappa$ for every large enough κ; $\boldsymbol{\mathcal{U}}$ is then metrizable and complete. We finally let $\boldsymbol{\mathcal{U}}_K$ be the subset of $\mathbf{u} \in \boldsymbol{\mathcal{U}}$ taking values almost everywhere in $\mathcal{H}_K$—topologized through $\|\cdot\|_\kappa$ with $\kappa := \|K\|$.

Finally, we introduce the map

$$\mathbf{X} : \mathbf{u} \mapsto X := \begin{pmatrix} 0 & -z \\ \bar{z} & 0 \end{pmatrix} \quad \text{for } \mathbf{u} = (u, v) \in \mathcal{H} \tag{2.4}$$

with $z = z(\cdot)$ defined by (2.2). Then (1.1.i, ii) can be written as an abstract ordinary differential equation with respect to ξ in the more succinct form

$$\mathbf{u}' = \mathbf{X}(\mathbf{u})\mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{H}. \tag{2.5}$$

[2] Clearly it would be sufficient to have this integrability only "to the left," i.e., on each semi-infinite interval $(-\infty, \tau]$ with τ finite, imposing no growth condition as $\tilde{\tau} \to +\infty$. The present formulation permits us to work with a Hilbert space formulation for $\mathbf{u} = (u, v)$ and the greater generality can be recovered—observing that, by setting everything equal to zero for $\tau > \tau^*$, we can always restrict the problem to $(-\infty, \tau^*]$ for each (arbitrary) finite τ^* to make the present formulation appropriate.

Note that although we use a $\mathbb{C}^2$ notation, thinking of $\mathcal{H}$ as a complex Hilbert space, it will later be convenient (cf., Thm. 3.4) to treat it also as a real Hilbert space, effectively identifying $\mathbb{C}^2$ with $\mathbb{R}^4$.

3. Well posedness. Our principal concern here is to show that the problem (2.5) has a unique solution, but we begin with a lemma about the map $\mathbf{u} \mapsto \mathbf{X}(\mathbf{u})$.

LEMMA 3.1. *The map* $\mathbf{X}$ *is well defined by* (2.2) *and* (2.4), *and is continuous from* $\mathcal{H}$ *to* $\mathcal{X}$. *For each* $\tilde{\mathcal{H}}_K := \cup\{\mathcal{H}_{\tilde{K}} : \tilde{K} \leq K\}$, *the set of functions* z *defined by* (2.2) *with* $\mathbf{u} \in \tilde{\mathcal{H}}_K$ *is precompact in* $\mathcal{Z}$ *and the map* $\mathbf{X}$ *is uniformly Lipschitzian on* $\tilde{\mathcal{H}}_K$ *with Lipschitz constant* $\kappa := \|K\|$.

Proof. Suppose $\mathbf{u} \in \mathcal{H}$ with $|\mathbf{u}(\tau)| \leq K(\tau)$ almost everywhere and obtain z from $\mathbf{u}$ as in (2.2). Noting that $e^{-\gamma(\tau-\tilde{\tau})} \leq 1$ and that $2|u\overline{v}| \leq K^2$, we then clearly have

$$|z(\tau) - z(\hat{\tau})| = \left| \int_{\hat{\tau}}^{\tau} e^{-\gamma(\tau-\tilde{\tau})} u\overline{v} \right| \leq \tfrac{1}{2} \int_{\hat{\tau}}^{\tau} K^2$$

which shows the continuity of z and, indeed, equicontinuity on $\tilde{\mathcal{H}}_K$; essentially the same computation shows that $\{z(\tau_n)\}$ is (uniformly on $\tilde{\mathcal{H}}_K$) always Cauchy as $\tau_n \to \infty$ so $z(\tau)$ always has a limit as $\tau \to \infty$. Similarly, there is a uniform bound: $|z(\tau)| \leq \frac{1}{2}\kappa^2$. By the Arzela–Ascoli Theorem, it follows that the relevant $\{z\}$ will be in a compact subset of $\mathcal{Z}$. Now let z_1, z_2 be obtained from $\mathbf{u}_1, \mathbf{u}_2$ and set $z := z_1 - z_2$, $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$ so, pointwise in τ, we have

$$\begin{aligned} &u_1\overline{v}_1 - u_2\overline{v}_2 = u_1\overline{v} + u\overline{v}_2 = u\overline{v}_1 + u_2\overline{v}, \\ &|u_1\overline{v}_1 - u_2\overline{v}_2| \leq \min\{[|u_1|^2 + |v_2|^2],\ [|v_1|^2 + |u_2|^2]\}^{1/2}|\mathbf{u}|. \end{aligned}$$

Note that this minimum is bounded by the average—which is bounded by $K(\tau)$ for $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{H}_K$. Thus,

$$|z(\tau)| \leq \int_{-\infty}^{\tau} K|\mathbf{u}| \leq \|K\|\|\mathbf{u}\|.$$

Since we are using $\sup\{|z(\cdot)|\}$ as our $\mathcal{X}$-norm, we then get for $\mathbf{X}(\cdot)$ the desired Lipschitz condition with constant $\kappa := \|K\|$. □

THEOREM 3.2. *Let* $\mathbf{u}_0 = (u_0, v_0)$ *be given in* $\mathcal{H}$. *Then there is a unique function* $\mathbf{u} = (u, v) : \mathbb{R}_+ \to \mathcal{H}$ *in* $\boldsymbol{\mathcal{U}}_K$ *satisfying the nonlinear equation* (2.5) *with the notation of* (2.1) *and* (2.4).

Before beginning the proof, we remark that an essentially identical argument works for the problem with ξ reversed (here and also in Theorem 3.3) so, in particular, we have *backward uniqueness* for the solution as well. We also remark that our definition of $\boldsymbol{\mathcal{U}}_K$ means that finding $\mathbf{u} \in \boldsymbol{\mathcal{U}}_K$ implicitly includes the assertion of (1.4).

Proof. Fix $\mathbf{u}_0 \in \mathcal{H}$, thus fixing $K := |\mathbf{u}| \in L^2_+(\mathbb{R})$ and the spaces $\mathcal{H}_K, \boldsymbol{\mathcal{U}}_K$ as above. There is no difficulty in defining a map $\mathbf{F} : \tilde{\mathbf{u}} \mapsto \mathbf{u}$ for $\tilde{\mathbf{u}} \in \boldsymbol{\mathcal{U}}_K$ by solving the linear ordinary differential equation

$$\mathbf{u}' = \mathbf{X}(\tilde{\mathbf{u}})\mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{3.1}$$

Indeed, we note from Lemma 3.1 that $\mathbf{X}(\tilde{\mathbf{u}})\mathbf{u}$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$ so (3.1) can be interpreted pointwise in τ as a (finite-dimensional) ordinary differential equation in ξ—with an adequately defined initial condition for almost every τ. Note that, since $X := \mathbf{X}(\tilde{\mathbf{u}})$ is skew-adjoint, we have from (3.1) that

$$(|\mathbf{u}|^2)' = 2\langle \mathbf{u}, \mathbf{u}' \rangle = 2\langle \mathbf{u}, X\mathbf{u} \rangle \equiv 0$$

whence $|\mathbf{u}(\cdot,\tau)|$ is constant and we have (1.4) for solutions of (3.1), i.e., we have $\mathbf{u}(\cdot) \in \boldsymbol{\mathcal{U}}_K$. (Indeed, we need not even have $\tilde{\mathbf{u}} \in \boldsymbol{\mathcal{U}}_K$ to have $\mathbf{u} =: \mathbf{F}(\tilde{\mathbf{u}}) \in \boldsymbol{\mathcal{U}}_K$ and $\|\mathbf{u}(\xi,\cdot)\| \equiv \kappa$.)

A fixed point for $\mathbf{F}$ is a solution of (2.5) so it will be sufficient to show that $\mathbf{F}$ is a uniformly strict contraction from $\boldsymbol{\mathcal{U}}_K$ to itself. Given $\mathbf{u}_j := \mathbf{F}(\tilde{\mathbf{u}}_j)$ for $j = 1, 2$, set $\mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$, $\tilde{\mathbf{u}} := \tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2$ and let $\tilde{\nu}$ be the $\boldsymbol{\mathcal{U}}$-norm of $\tilde{\mathbf{u}}$. Then

$$\begin{aligned}(e^{-2\kappa^2\xi}\|\mathbf{u}\|^2)' &= e^{-2\kappa^2\xi}\left(-2\kappa^2\|\mathbf{u}\|^2 + 2\langle \mathbf{u}, [\mathbf{X}(\tilde{\mathbf{u}}_1) - \mathbf{X}(\tilde{\mathbf{u}}_2)]\,\mathbf{u}_2\rangle\right)\\ &\le e^{-2\kappa^2\xi}\left(-2\kappa^2\|\mathbf{u}\|^2 + 2\kappa\|\tilde{\mathbf{u}}\|\|\mathbf{u}_2\|\right)\\ &\le (\kappa^2/2)\, e^{-2\kappa^2\xi}\|\tilde{\mathbf{u}}\|^2.\end{aligned}$$

Since $\|\tilde{\mathbf{u}}(\xi)\|^2 \le \exp[4\kappa^2\xi]\tilde{\nu}^2$ and $\mathbf{u}(0) = 0$, integrating gives

$$e^{-2\kappa^2\xi}\|\mathbf{u}\|^2 \le \frac{\kappa^2}{2}\int_0^\xi e^{2\kappa^2\bar{\xi}} \le \tfrac{1}{4}e^{2\kappa^2\xi}\tilde{\nu}^2$$

and then

$$\left(e^{-2\kappa^2\xi}\|\mathbf{u}\|\right)^2 \le \tilde{\nu}^2/4$$

which shows that $\mathbf{F}$ is uniformly Lipschitzian on $\boldsymbol{\mathcal{U}}_K$ with Lipschitz constant $\frac{1}{2}$. The result then follows by the Contraction Mapping Theorem. □

We complete our treatment of well posedness by considering the continuous dependence of the solution on the initial data $\mathbf{u}_0$. It is clear that the estimate (3.2) gives continuous dependence of solutions on initial data in the sense of uniform convergence (with respect to the $\mathcal{H}$-norm) on bounded ξ-intervals but the estimate grows exponentially in ξ. Since we know that the solutions themselves are bounded uniformly in ξ, it might seem plausible that this could be improved to have convergence uniform on $\mathbb{R}_+$. That, however, is false; see Remark 6.4.

THEOREM 3.3. *Let* $\mathbf{u}_{j0} = (u_{j0}, v_{j0})$ *for* $j = 1, 2$ *be given in* $\mathcal{H}$ *with corresponding solutions* $\mathbf{u}_j : \mathbb{R}_+ \to \mathcal{H}$ *satisfying* (2.5). *Then*

$$\|\mathbf{u}_1(\xi,\cdot) - \mathbf{u}_2(\xi,\cdot)\| \le e^{(\kappa_+\kappa_-)\xi}\|\mathbf{u}_{10} - \mathbf{u}_{20}\| \tag{3.2}$$

where $\kappa_\pm := \|K_\pm\|$ *with* $K_\pm(\tau) := \max, \min\{|\tilde{\mathbf{u}}_0(\tau)|, |\hat{\mathbf{u}}_0(\tau)|\}$.

Proof. The argument is standard. Set $\mathbf{v} := \mathbf{u}_1 - \mathbf{u}_2$ and $X := X_1 - X_2$ with $X_j := \mathbf{X}(\mathbf{u}_j)$, etc. Pointwise in τ, we have $|\mathbf{v}|^2 = \langle \mathbf{v}, \mathbf{v}\rangle$ so

$$\begin{aligned}\partial|\mathbf{v}|^2/\partial\xi &= 2\mathrm{Re}[\langle \mathbf{v}, X_1\mathbf{v}\rangle + \langle \mathbf{v}, X\mathbf{u}_2\rangle]\\ &= 2\mathrm{Re}[\langle \mathbf{v}, X_2\mathbf{v}\rangle + \langle \mathbf{v}, X\mathbf{u}_1\rangle]\\ &= 2\mathrm{Re}\langle \mathbf{v}, X\mathbf{u}_2\rangle = 2\mathrm{Re}\langle \mathbf{v}, X\mathbf{u}_1\rangle\end{aligned}$$

by the skew symmetry of X_1, X_2. By Lemma 3.1, we have $|X(\tau)| \le \|K_+\|\,\|\mathbf{v}\|$ and (1.4) gives, pointwise in τ, $\min\{|\mathbf{u}_1|, |\mathbf{u}_2|\} \equiv K_-$. Thus,

$$\begin{aligned}\partial|\mathbf{v}|^2/\partial\xi &\le 2|\mathbf{v}(\tau)|\|K_+\|\|\mathbf{v}\|K_-(\tau),\\ d\|\mathbf{v}\|^2/d\xi &\le 2\|\mathbf{v}\|^2\|K_+\|\|K_-\|.\end{aligned}$$

The result now follows on applying the Gronwall inequality. □

Extending Theorem 3.3, we next wish to consider linearization of the system, i.e., differentiability of the dependence on initial data.

THEOREM 3.4. *The solution map* $\mathbf{S} : \mathcal{H} \to \boldsymbol{\mathcal{U}}$ *for* (2.5), *is "Fréchet differentiable" (in a sense to be made precise below). At each* $\mathbf{u}_0^* \in \mathcal{H}$ *and corresponding* $\mathbf{u}^* := \mathbf{S}(\mathbf{u}_0^*)$ *the derivative is the linear map:* $\mathcal{H} \to \boldsymbol{\mathcal{U}} : \mathbf{u}_0 = (u_0, v_0) \mapsto \mathbf{u}$ *defined by the linearized system:*

$$(3.3) \qquad \begin{cases} u' &= -z^* v - v^* z, \\ v' &= \bar{z}^* u + u^* \bar{z}, \end{cases} \qquad z := \int_{-\infty}^{\tau} e^{-\gamma(\tau - \tilde{\tau})} [\bar{v}^* u + u^* \bar{v}] \, d\tilde{\tau}$$

with $\mathbf{u}(0) = \mathbf{u}_0$. *In particular, for any* $\mathbf{u}_0^* \in \mathcal{H}$ *yielding* $\mathbf{u}^*$ *by* (2.5) *the equation* (3.3) *has a unique solution* $\mathbf{u}$ *for each initial* $\mathbf{u}_0 \in \mathcal{H}$ *and* $\mathbf{u}(\xi, \cdot)$ *will be bounded in* $\mathcal{H}$ *uniformly on bounded* ξ*-intervals.*

A word of caution is in order here since we have been working with complex spaces: *the solution map is not differentiable when complex differentiation is considered* since (2.5) involves conjugations. Instead, as noted earlier, although we have made no alteration in the notation, we are here considering $\mathcal{H} = L^2(\mathbb{R} \to \mathbb{C}^2)$ as isometrically equivalent to the *real* Hilbert space $L^2(\mathbb{R} \to \mathbb{R}^4)$, etc.

Proof. Now consider Theorem 3.3 with $\mathbf{u}_{20} = \mathbf{u}_0^*$ and, for $s \neq 0$, $\mathbf{u}_{10} = \mathbf{u}_0^* + s\mathbf{u}_0$; set $\mathbf{v} = \mathbf{v}(\xi, \tau; s) := [\mathbf{u}_1 - \mathbf{u}^*]/s$. Then (3.2) gives the uniform estimate $\|\mathbf{v}\| \leq e^{(\kappa_+ + \kappa_-)\xi} \|\mathbf{u}_0\|$, where $\kappa_\pm = \kappa_\pm(s) \to \|\mathbf{u}_0^*\|^2$ as $s \to 0$. A standard argument then shows that $\mathbf{v}(\cdot; s)$ satisfies a system whose right-hand side tends to that of (3.3) as $s \to 0$ ($\mathcal{O}(s)$ difference in the coefficients) so $\|\mathbf{v}(\cdot; s) - \mathbf{u}\|_\kappa \to 0$ for any $\kappa > \|\mathbf{u}_0^*\|$. Temporarily fixing any such κ, we may treat (the relevant subspace of) $\boldsymbol{\mathcal{U}}$ as a Banach space with the norm $\|\cdot\|_\kappa$ and $\mathbf{u}$, given by (3.3), is the Gâteaux differential of $\mathbf{S}(\cdot)$ at $\mathbf{u}^*$ in the direction of $\mathbf{u}_0$. This is clearly linear in $\mathbf{u}_0$, so this gives a Gâteaux derivative. It is continuous in $\mathbf{u}_0^*$ (as long as we stay close enough to the original $\mathbf{u}_0^*$ so as not to disturb the choice of κ), so this is necessarily a Fréchet derivative, working with this $\|\cdot\|_\kappa$. [We do note that when considering bounded ξ-intervals, the choice of κ is irrelevant; in any case, (3.2) gives control of errors in the norm with $\kappa = \|\mathbf{u}_0^*\|$.] □

4. Some remarks. Our first concern here is to verify (1.5).

LEMMA 4.1. *Let* $\mathbf{u}$ *be any solution of the system* (2.5) *with* $\mathbf{u}_0$ *in* L^2. *Then,*

$$(4.1) \qquad \begin{cases} \text{(i)} & \left(\int_{-\infty}^{\tau} |e^{-\gamma(\tau - \tilde{\tau})} u|^2\right)' = -|z|^2, \\ \text{(ii)} & \left(\int_{-\infty}^{\tau} |e^{-\gamma(\tau - \tilde{\tau})} v|^2\right)' = |z|^2, \\ \text{(iii)} & z'(\cdot, \tau) = \int_{-\infty}^{\tau} e^{-\gamma(\tau - \tilde{\tau})} \left(|u|^2 - |v|^2\right) z \end{cases}$$

for all $\xi > 0$ *and all* $\tau \in \mathbb{R}$.

Proof. From (1.1) we have $[e^{\gamma\tau} z]_\tau{}' = e^{\gamma\tau} \left(|u|^2 - |v|^2\right) z$ and, using (4.5), integrating this gives (4.1.iii). Similarly, we have

$$\overline{e^{\gamma\tau} z}(e^{\gamma\tau} z)_\tau = e^{2\gamma\tau} \bar{z}(u\bar{v}) = e^{2\gamma\tau} \bar{v} v_\xi = -e^{2\gamma\tau} u \overline{u_\xi}$$

and integrating this gives (4.1.i) and (4.1.ii). That these identities hold pointwise for *all* $\tau \in [0, 1]$ follows from the known continuity in τ of z and continuity of the indefinite integral in the third identity. □

COROLLARY 4.2. *If z is real (alternatively, if z is pure imaginary or if z vanishes identically) for $\tau \le \tau_*$ at $\xi = \xi_0$, then this holds for all $\xi \ge 0$. For the case in which $z \equiv 0$ on $\mathbb{R}_+ \times (-\infty, \tau_*)$, we have $\mathbf{u}$ stationary (independent of ξ) there and conversely.*

Proof. The first assertion follows immediately from (4.1.iii), viewing $e^{-\gamma(\tau-\tilde{\tau})}(|u|^2 - |v|^2)$ simply as an integrable real function and integrating this ODE forward or, as in the remark following the statement of Theorem 3.2, backward in ξ. The case of $z \equiv 0$ is obvious with the converse following by, e.g., (4.1.i). □

Remark 4.3. While the system (1.1) cannot give analyticity in its dependence on the initial data, we observe that we could consider the analytic system

$$(4.2)\qquad \begin{cases} u_1' = -z_1 v_1, \\ v_1' = z_2 u_1, \\ u_2' = -z_2 v_2, \\ v_2' = z_1 u_2, \end{cases} \qquad \begin{cases} z_1 := \int_{-\infty}^{\tau} e^{-\gamma(\tau-\tilde{\tau})} u_1 v_2, \\ z_2 := \int_{-\infty}^{\tau} e^{-\gamma(\tau-\tilde{\tau})} u_2 v_1, \end{cases}$$

and have

$$(4.3)\qquad [u_1, v_1, z_1] \equiv [u, v, z] \qquad [u_2, v_2, z_2] \equiv [\overline{u}, \overline{v}, \overline{z}]$$

for all real $\xi > 0$ if this holds initially, at $\xi = 0$.

Without (4.3), we do not have the estimate (1.4) and so it is not clear when solutions for (4.2) will exist globally. On any finite ξ-interval, however, we can get existence for initial data almost satisfying (4.3) and thus, analytic linearization, subject to (4.3), with (3.3) suitably extended to a complex neighborhood of $\mathbb{R}_+$. In particular, this shows that we can also obtain higher derivatives of the solution map with respect to the initial data.

Remark 4.4. For nonstationary solutions, it is interesting to consider the case in which $\gamma = 0$ and z is real on $\mathbb{R}$ — initially, at $\xi = 0$, and so for all $\xi \ge 0$ by Corollary 4.2.

For this we will first reduce the problem to a more convenient form, modifying our notation somewhat, without (at first) taking $\gamma = 0$. If we set $\mathbf{u} =: K\tilde{\mathbf{u}}$ pointwise in τ, then we have the identity

$$(4.4)\qquad |\tilde{\mathbf{u}}|^2 := |\tilde{u}|^2 + |\tilde{v}|^2 \equiv 1 \quad \text{pointwise in } \xi, \tau$$

in view of Theorem 3.2. It is easily seen that $\tilde{\mathbf{u}}$ also satisfies (2.5), provided we modify the definition of the operator $\mathbf{X}(\cdot)$ by replacing (2.2) with

$$(4.5)\qquad z(\tau) := \int_{-\infty}^{\tau} e^{-\gamma(\tau-\tilde{\tau})} u(\tilde{\tau}) \overline{v(\tilde{\tau})} K^2(\tilde{\tau})\, d\tilde{\tau}.$$

Of course, the initial data now must satisfy: $|\tilde{\mathbf{u}}_0| \equiv 1$, pointwise in τ.

That much reduction is available for all γ, but when $\gamma = 0$ we can conveniently use the variable σ of (2.1) to view $\tilde{\mathbf{u}}$ as a function of (ξ, σ), rather than of (ξ, τ). This further reduction will actually (σ—almost everywhere by Sard's Theorem) avoid any difficulty with the definition of $\tilde{u}$ when $K(\tau) = 0$. We note that it is possible to view z also as a function of (ξ, σ) since, while the function $\sigma(\cdot)$ may not be injective, this can happen only if K vanishes on some subinterval — in which case $\mathbf{u}$ (whence $u\overline{v} = z_\tau$)

also vanishes on this subinterval so z is constant there: the value of $z(\xi,\tau)$ depends only on (ξ,σ). In this case, the domain of σ is $[0,\kappa^2]$ and, henceforth omitting the $\tilde{}$, (4.5) becomes simply

$$z(\cdot,\sigma) := \int_0^\sigma u\bar{v}\,d\tilde{\sigma}. \tag{4.6}$$

Apart from the name of the variable ($\sigma \leftrightarrow \tau$), we observe that this is identical to the *original* problem for $\gamma = 0$ with initial data giving

$$K(\tau) = \{1 \text{ for } 0 \le \tau \le \kappa^2; 0 \text{ else}\}.$$

Note that use of (4.6) means that we have the boundary condition

$$z = 0 \quad \text{at } \sigma = 0, \tag{4.7}$$

corresponding to (1.2).

In view of (4.4), $\mathbf{u}$ must have the form

$$u = e^{i\vartheta}\cos\varphi, \qquad v = e^{i\vartheta}\sin\varphi \tag{4.8}$$

with ϑ,φ real. Assume ϑ is independent of ξ — first as an *ansatz*, but then confirmed by our subsequent calculations. We then see that (2.5), (4.6) are equivalent to the requirement that $z = \varphi_\xi$ and

$$\varphi_{\xi\sigma} = u\bar{v} = \tfrac{1}{2}\sin 2\varphi. \tag{4.9}$$

If we set $t := \sigma + \xi$ and $x := \sigma - \xi$, then (4.9) becomes

$$2\varphi_{tt} - 2\varphi_{xx} = \sin 2\varphi, \tag{4.10}$$

i.e., 2φ satisfies the sine-Gordon equation. Conversely, if $2\varphi = 2\varphi(t,x)$ is any real solution of the sine-Gordon equation, then, for arbitrary real $\vartheta(\sigma)$,

$$\begin{aligned} u(\xi,\sigma) &:= e^{i\vartheta(\sigma)}\cos\varphi(\sigma+\xi,\sigma-\xi),\\ v(\xi,\sigma) &:= e^{i\vartheta(\sigma)}\sin\varphi(\sigma+\xi,\sigma-\xi),\\ z &:= [\varphi_t - \varphi_x](\sigma+\xi,\sigma-\xi) \end{aligned}$$

gives a solution of (1.1) for $\gamma = 0$ with $z \equiv \varphi_\xi$ real.

For this to be consistent with the boundary conditions (4.7) we are imposing on z, it is necessary that 2φ be a solution of the sine-Gordon equation satisfying

$$\varphi_t(t,x) \equiv \varphi_x(t,x) \quad \text{along the line: } t + x = 0. \tag{4.11}$$

While the sine-Gordon equation has nontrivial traveling wave solutions, we emphasize that those are all excluded by this constraint (4.11)—corresponding to (4.7) and so to our original boundary condition (1.2) at $\tau \to -\infty$.

5. Discrete approximation. In this section we consider the "obvious" discretizations (with respect to τ) of the system (2.5). For convenience, we restrict our attention to the case $\gamma = 0$ and take the system reduced as in Remark 4.4 so that $|u_0|^2 + |v_0|^2 \equiv 1$ for $0 \le \sigma \le \kappa^2$ and

$$\begin{cases} u' &= -zv \\ v' &= \bar{z}u \end{cases} \qquad \text{with } z(\cdot,\sigma) := \int_0^\sigma u\bar{v}\,d\tilde{\sigma}. \tag{5.1}$$

While much of our analysis here directly parallels that for the partial differential equation system, for this finite-dimensional approximation we have the advantage of local compactness and will be able to obtain a more complete description of the asymptotic behavior of solutions as $\xi \to \infty$. This may be viewed both as a theoretical complement to the results observed in computational simulation and for comparison with our less complete asymptotic analysis in the next section, as an indication of goals and conjectures for future work.

For the remainder of this section, we adopt the notation that

$$\begin{aligned} \mathcal{H} &:= (\mathbb{C}^2)^J, \\ \mathbf{u} &:= (\mathbf{u}_1, \cdots, \mathbf{u}_J) \in \mathcal{H}, \qquad [\mathbf{u}_j := (u_j, v_j) \in \mathbb{C}^2], \\ \mathcal{H}_1 &:= \{\mathbf{u} \in \mathcal{H} : |\mathbf{u}_j|^2 := |u_j|^2 + |v_j|^2 = 1\}, \\ \mathcal{S} &:= \{\mathbf{u}^0 \in \mathcal{H}_1 : u_j^0 = 0 \text{ or } v_j^0 = 0 \quad \text{for each } j = 1, \cdots, J, \\ U_j &:= \delta \Sigma_1^j |u_k|^2, \\ \nu &:= \max\{U_j / j\delta : j = 1, \cdots, J\}. \end{aligned}$$

In general, we have $U_j = U_j(\cdot)$ for some particular solution $\mathbf{u}(\cdot)$ of (5.3), below, and ν will similarly relate to this; we could write explicitly $\nu := \nu(\mathbf{u})$ for $\mathbf{u} \in \mathcal{H}_1$ or $\nu := \nu(\xi; \mathbf{u}^0) := \nu(\mathbf{u}(\xi))$, with $\mathbf{u}(\cdot)$ satisfying (5.3) with initial data $\mathbf{u}^0$.

Assuming the nodes are equally spaced with respect to σ, we can then define $z_j \approx z(j\delta)$ by a discretized approximation to the integral

$$z_j := \delta \Sigma_1^j u_k \overline{v_k} = \delta u_j \overline{v_j} + z_{j-1}, \qquad z_0 := 0, \tag{5.2}$$

for $j = 1, \cdots, J$. Thus, we consider here the system of ordinary differential equations

$$\begin{aligned} u_j' &= -z_j v_j \\ v_j' &= \overline{z_j} u_j \\ \text{with } u_j(0) &= u_j^0, \quad v_j(0) = v_j^0 \end{aligned} \tag{5.3}$$

for each $j = 1, \cdots, J$, using (5.2). For the initial conditions we assume, as earlier, that

$$|u_j^0|^2 + |v_j^0|^2 = 1 \quad \text{for } j = 1, \cdots, J.$$

The factor δ could be removed by rescaling ξ, but we will retain $\delta := \kappa^2/J$ here to remind us of the correspondence: $u_j(\xi) \approx u(\xi, j\delta)$, etc. We note that (5.3) is equivalent to

$$\begin{aligned} u_j' &= -\delta |v_j|^2 u_j - \delta \zeta_j v_j, \\ v_j' &= \delta |u_j|^2 v_j + \delta \zeta_j u_j \end{aligned} \tag{5.4}$$

with $\delta \zeta_j := z_{j-1}$.

We begin by asserting the set of "background" results.

LEMMA 5.1. *For each $\mathbf{u}^0 \in \mathcal{H}$ there is a unique global solution $\mathbf{u}(\cdot) = \mathbf{u}(\cdot; \mathbf{u}^0)$ of (5.3), depending on $\mathbf{u}^0$ uniformly on bounded ξ-intervals. The functions $u_j, \overline{u}_j, \cdots, \overline{z}_j$ are all real-analytic functions of ξ. For $\mathbf{u}^0 \in \mathcal{H}_1$, we have $\mathbf{u}(\xi) \in \mathcal{H}_1$ for all $\xi \geq 0$.*

Proof. The arguments are straightforward parallels of those given above for Theorems 3.2, 3.4, and Remark 4.3. Details are left to the reader. □

LEMMA 5.2. *For $j = 1, \cdots, J$ we have*

$$(5.5) \qquad \begin{cases} \text{(i)} & [u_j\overline{v}_j]' = z_j\left[|u_j|^2 - |v_j|^2\right], \\ \text{(ii)} & z_j' = \delta\Sigma_1^j z_k\left[|u_k|^2 - |v_k|^2\right], \\ \text{(iii)} & -U_j' = |z_j|^2 + \delta^2\Sigma_1^j|u_k\overline{v}_k|^2. \end{cases}$$

If, for $k = 1, \cdots, j$, we have $z_k(0) = 0$, then this persists for all ξ, and u_k, v_k are then constant (independent of ξ).

Proof. The formulas (5.5.i, ii) are direct from (5.3) and the final assertion follows; compare Corollary 4.2. Also from (5.3), for each k we have

$$\begin{aligned} -\delta(|u_k|^2)' &= \delta[\overline{u}_k(z_kv_k) + u_k\overline{z_kv_k}] \\ &= \overline{(z_k - z_{k-1})}z_k + \overline{z}_k(z_k - z_{k-1}) \\ &= |z_k|^2 - |z_{k-1}|^2 + |z_k - z_{k-1}|^2, \end{aligned}$$

since $z_k - z_{k-1} = \delta u_k\overline{v}_k$. Summing over $k = 1, \cdots, j$ gives (5.5.iii). □

LEMMA 5.3. *If (for some k) z_k is not identically zero, then (for each $j \geq k$) z_j can vanish at most on a discrete set (with no finite limit points) and U_j is strictly decreasing on every ξ-interval.*

Proof. By the real-analyticity noted in Lemma 5.1, it is only possible for z_j to vanish on a set with a finite limit point if $z_j \equiv 0$; else $|z_j|^2 > 0$ almost everywhere and (5.5) implies U_j strictly decreasing. To have $z_j \equiv 0$ we must either have $u_j \equiv 0$ or $v_j \equiv 0$; suppose the former, so $|v_j| \equiv 1$. Since (5.4) would then give $0 \equiv u_j' = -z_{j-1}v_j$, this is only possible if also $z_{j-1} \equiv 0$. Similarly, $v_j \equiv 0$ would also require $z_{j-1} \equiv 0$. Induction on the index completes the proof. □

LEMMA 5.4. *For arbitrary initial data in $\mathcal{H}_1$, we have each $z_j \to 0$ and $\mathbf{u} \to \mathcal{S}$ as $\xi \to \infty$.*

Proof. Since U_J is nonincreasing by (5.5) and is obviously bounded below by zero, we must have $[|z_j|^2 + \delta^2\Sigma_1^j|u_k\overline{v}_k|^2]$ integrable. As we know that u_j, v_j, z_j are bounded, this has a bounded derivative and so must go pointwise to zero. In particular, this immediately gives $z_j \to 0$. Each component of $\mathbf{u}$ lives in the (compact) unit sphere of $\mathbb{C}^2$ so, setting $\varphi(u,v) := |u\overline{v}|$, convergence $\varphi(u,v) \to 0$ implies that $(u,v) \to \varphi^{-1}(\{0\}) = \{(u,v) \in \mathcal{H}_1 : u = 0 \text{ or } v = 0\}$. Thus, $\mathbf{u} \to \mathcal{S} = \{\text{each } u_j = 0 \text{ or } v_j = 0\}$ as asserted. □

LEMMA 5.5. *Let $\mathbf{u}(\cdot)$ be a solution of (5.3) such that $U_J(\xi_*) < \delta$ for some $\xi_* \geq 0$. Then $\mathbf{u}(\tilde{\xi}) \in \mathcal{I}_0$ for each $\tilde{\xi} \geq 0$, where*

$$(5.6) \qquad \mathcal{I}_0 := \{\mathbf{u}^0 \in \mathcal{H}_1 : \textit{each } u_j \to 0 \textit{ as } \xi \to \infty\}.$$

The set $\mathcal{I}_0$ is open in $\mathcal{H}_1$.

Proof. For each j, the assumption precludes having $|v_j| \to 0$ since U_J is nonincreasing and $|u_j| = 1$ implies $U_J \geq \delta$; by Lemma 5.4, this ensures $u_j \to 0$. By the definition, this gives $\mathbf{u}^0 \in \mathcal{I}_0$ and, of course, each $\mathbf{u}(\tilde{\xi} \in \mathcal{I}_0$. To see that $\mathcal{I}_0$ is open, consider any $\tilde{\mathbf{u}}^0 \in \mathcal{I}_0$ so each $\tilde{u}_j \to 0$. We may then find ξ_* such that $\tilde{U}_J(\xi_*) < \delta/2$. By Lemma 5.1 we have uniform continuity on bounded ξ-intervals for the dependence of solutions of (5.3) on the initial data, so there is a neighborhood of $\tilde{\mathbf{u}}^0$ giving $U_J(\xi_*) < \delta$. The first part of this proof then shows this neighborhood is in $\mathcal{I}_0$. □

THEOREM 5.6. *For arbitrary initial data in* $\mathcal{H}_1$ *we have convergence* $\mathbf{u} \to \mathbf{u}^*$ *(at an asymptotically exponential rate) as* $\xi \to \infty$ *for some steady state solution* $\mathbf{u}^* \in \mathcal{S}$. *Thus, for each* $j = 1, \cdots, J$ *we have either* Case 1: $u_j \to 0$ *and* $v_j \to v_j^*$ *(with* $|v_j^*| = 1$*) or* Case 2: $v_j \to 0$ *and* $u_j \to u_j^*$ *(with* $|u_j^*| = 1$*); in particular, we always have* Case 1 *for* $j = 1$.

Proof. We will proceed inductively in j, taking the system in the form (5.4) with the index j suppressed so

$$\text{(5.7)} \qquad \begin{cases} \text{(i)} & u' = -\delta(|v|^2 u + \zeta v), \\ \text{(ii)} & v' = \ \delta(|u|^2 v + \overline{\zeta} u), \end{cases}$$

and with the inductive assumption that we know

$$\text{(5.8)} \qquad |\zeta(\xi)| \le C e^{-\mu\delta\xi}$$

for $C = C_\mu$ and arbitrary $0 < \mu < 1$. By Lemma 5.4 we know that $u\overline{v} \to 0$ as $\xi \to \infty$, so we must be in one of the two possible cases—which we then consider separately to show the exponential decay rate for the appropriate component. With (4.4) and the inductive hypothesis on $\zeta = \zeta_j$, this completes the induction by giving the corresponding exponential decay for ζ_{j+1}. Returning to (5.3) with knowledge of exponential decay of z_j, integrability of the derivative gives existence of a specific limit for the nonvanishing component as well.

Case 1 $\boxed{u \to 0}$. Fix $\mu < 1$. Set $y := |u|^2$ so $y' = 2\,\mathrm{Re}\,\overline{u}u' = -2\delta[|v|^2 y + \mathrm{Re}\,\zeta\overline{u}v$, using (5.7.i) and note that $\mathrm{Re}\,\zeta\overline{u}v \le |\zeta||v|\sqrt{y} \le \varepsilon|v|^2 y + |\zeta|^2/4\varepsilon$. Now choose $0 < \varepsilon < 1 - \mu$ and use (5.8) with μ replaced by $\tilde{\mu} := \mu + \varepsilon < 1$. By (4.4), if $u \to 0$, then $|v| \to 1$ so (noting that $\mu/(1-\varepsilon) < 1$) there exists $\tilde{\xi}$ such that $(1-\varepsilon)|v(\xi)|^2 \ge \mu$ for $\xi \ge \tilde{\xi}$. Thus, we have

$$y'(\xi) \le -2\delta\mu y + (\delta C^2/2\varepsilon)e^{-2(\mu+\varepsilon)\delta\xi}$$

for $\xi \ge \tilde{\xi}$. The Gronwall inequality and some simple manipulation then give the desired estimate for $y = |u|^2$:

$$\begin{aligned} |u(\xi)|^2 &\le e^{-2\mu\delta(\xi-\tilde{\xi})}|u(\tilde{\xi})|^2 + \frac{\delta C^2}{2\varepsilon}\int_{\tilde{\xi}}^{\xi} e^{-2\mu\delta(\xi-\hat{\xi})}e^{-2(\mu+\varepsilon)\hat{\xi}}\,d\hat{\xi} \\ &\le \left[e^{2\mu\delta\tilde{\xi}}|u(\tilde{\xi})|^2 + \frac{\delta C^2}{2\varepsilon}\int_{\tilde{\xi}}^{\infty} e^{-2\varepsilon\hat{\xi}}\,d\hat{\xi}\right] e^{-2\mu\delta\xi} \\ &=: \tilde{C}^2 e^{-2\mu\delta\xi} \end{aligned}$$

for $\xi \ge \tilde{\xi}$; this also applies to all $\xi \ge 0$ with a modification of the $\tilde{C}$.

Case 2 $\boxed{v \to 0}$. Again, fix $\mu < 1$ and now set $y := |v|^2$ and choose $0 < 2\varepsilon < 1 - \mu$. Much as above, we get

$$y'(\xi) \ge 2\delta(\mu - \varepsilon)y - (\delta C^2/2\varepsilon)e^{-2\mu\delta\xi}$$

for ξ large enough ($\xi \ge \tilde{\xi}_0$) that $(1-\varepsilon)|u(\xi)|^2 \ge \mu - \varepsilon$. Applying the (reversed)

Gronwall inequality we then obtain, for any $\xi > \tilde{\xi} \geq \tilde{\xi}_0$,

$$\begin{aligned} y(\xi) &\geq e^{2(\mu-\varepsilon)\delta(\xi-\tilde{\xi})} y(\tilde{\xi}) - \frac{\delta C^2}{2\varepsilon} \int_{\tilde{\xi}}^{\xi} e^{2(\mu-\varepsilon)\delta(\xi-\hat{\xi})} e^{-2\mu\hat{\xi}}\, d\hat{\xi} \\ &= e^{2(\mu-\varepsilon)\delta(\xi-\tilde{\xi})} \left[y(\tilde{\xi}) - \left(\frac{\delta C^2}{2\varepsilon} \int_{\tilde{\xi}}^{\xi} e^{-2\varepsilon\delta(\hat{\xi}-\tilde{\xi})}\, d\hat{\xi} \right) e^{-2\mu\delta\tilde{\xi}} \right] \\ &\geq e^{2(\mu-\varepsilon)\delta(\xi-\tilde{\xi})} \left[y(\tilde{\xi}) - \frac{C^2}{4\varepsilon} e^{-2\mu\delta\tilde{\xi}} \right]. \end{aligned}$$

It follows that we must have $|v(\tilde{\xi})| \leq (C/2\varepsilon)e^{-\mu\delta\tilde{\xi}}$ for all $\tilde{\xi} \geq \tilde{\xi}_0$ (hence, for all $\tilde{\xi} \geq 0$ with a modified coefficient) or the Gronwall estimate would give $y(\xi) \to \infty$, contradicting the case that $v \to 0$.

In view of Lemmas 5.3, 5.4, and 5.5 and the experience with computational experimentation with a variety of sets of initial data, it is plausible to conjecture that one always has $u_j \to 0$ as $\xi \to \infty$ unless $z_k \equiv 0$ for each $k = 1, \cdots, j$. This is false!

THEOREM 5.7. *Let* $\mathbf{u}^*$ *be any stationary solution such that* $u_j = 0$ *for some* $j < J$. *Then there is a nonstationary solution* $\mathbf{u}$ *such that* $\mathbf{u}(\xi) \to \mathbf{u}^*$ *as* $\xi \to \infty$.

Proof. For each j we have either:

Case 1 $\boxed{u_j^* = 0 \text{ so } |v_j^*| = 1}$. Set

$$u_j = v_j^* \sin\alpha_j, \quad v_j = v_j^* \cos\alpha_j, \quad \chi_j := -1$$

so $(v_j^* \cos\alpha_j)\alpha_j' = u_j' = -z_j v_j^* \cos\alpha_j$ or:

Case 2 $\boxed{v_j^* = 0 \text{ so } |u_j^*| = 1}$. Set

$$u_j = u_j^* \cos\alpha_j, \quad v_j = u_j^* \sin\alpha_j, \quad \chi_j := +1$$

so $(u_j^* \cos\alpha_j)\alpha_j' = v_j' = \overline{z_j} u_j^* \cos\alpha_j$.

In either case, we have $u_j\overline{v_j} = \frac{1}{2}\sin 2\alpha_j$ whence, after scaling ξ for convenience to permit the omission of a factor δ from our definition of z_j, we must have

$$\alpha_j' = \chi_j z_j, \qquad z_j = \Sigma_{k=1}^{j} \tfrac{1}{2} \sin 2\alpha_k \tag{5.9}$$

(provided $\cos 2\alpha_j \neq 0$) for $j = 1, \cdots, J$. Note that by restricting our attention here to the real case, we have isolated stationary solutions: each $\alpha_j = \pm\nu\pi$. Our notation ensures that any nonstationary solution $\alpha = [\alpha_1, \cdots, \alpha_J]$ of (5.9) for which $\alpha(\xi) \to 0$ corresponds to a nonstationary solution $\mathbf{u}$ of (5.4) for which $\mathbf{u} \to \mathbf{u}^*$ as $\xi \to \infty$. The linearization of (5.9) around $\alpha = 0$ is just

$$\alpha' = \mathbf{A}\alpha \quad \text{with } \mathbf{A} := \begin{pmatrix} \chi_1 & 0 & 0 & \cdots \\ \chi_2 & \chi_2 & 0 & \cdots \\ \chi_3 & \chi_3 & \chi_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.10}$$

Since $\mathbf{A}$ is lower triangular, we see that its eigenvalues (with multiplicity) are the diagonal elements $\{\chi_1, \cdots, \chi_J\}$ and if any of these is -1, we have a corresponding eigenvector giving an exponentially decaying solution. Since there are no purely imaginary eigenvalues, the linearized problem (5.10) gives the local splitting into stable and unstable manifolds whence the nonlinear problem (5.9) must also have an exponentially decaying solution near zero—asymptotically behaving precisely like the solution of (5.10). □

Remark 5.8. While we have carried through the analysis for the discretization corresponding to (5.2), we could equally well have considered a trapezoidal rule approximation to (4.6):

$$z_j := \delta\Sigma_1^j(u_{k-1}\overline{v_{k-1}} + u_k\overline{v_k})/2. \tag{5.11}$$

With trivial modification, we would then have obtained for that setting the same results obtained above. Indeed, (5.11) gives the *identical* system (5.4) if we were to scale ξ by 2 and set $\zeta_j := 2\sum_{k=1}^{j-1} u_k\overline{v_k}$, instead.

6. Stationary solutions. We now introduce the set $\mathcal{S}$ of all stationary solutions. Note that $\mathbf{u} \in \mathcal{S}$ means $\mathbf{u}' \equiv 0$ so $z \equiv 0$ on $\mathbb{R}$. By (2.2), this corresponds to having $u\overline{v} = 0$ almost everywhere; note from this that $\mathcal{S}$ is entirely independent of γ. It is easy to see that $\mathcal{S}$, $\mathcal{S}_0 := \{\mathbf{u} \in \mathcal{S} : u = u_0 \equiv 0\}$ and $\mathcal{S}\backslash\mathcal{S}_0 := \{\mathbf{u} \in \mathcal{S} : u_0 \not\equiv 0\}$ are each uncountable arc-wise connected sets in $\mathcal{H}_1$—even if we were to restrict attention to the "purely real" case, taking $\mathcal{H} := L^2(\mathbb{R} \to \mathbb{R}^2)$, or to factor out the action of the group $\mathcal{G} := \{g_\vartheta : \mathbf{u} \mapsto e^{i\vartheta(\tau)}\mathbf{u}\}$.

From the characterization $u\overline{v} \equiv 0$ we observe that, for each $\mathbf{u} \in \mathcal{S}$, we can partition[3] $\mathbb{R}$, independently of ξ in view of (1.4), as a disjoint union $\mathcal{A} \cup \mathcal{B}$ such that

$$\begin{cases} |u| = K, \ v = 0 & \text{on } \mathcal{A}, \\ u = 0, \ |v| = K & \text{on } \mathcal{B}. \end{cases} \tag{6.1}$$

We now wish to consider the linearization around a stationary solution $\mathbf{u}^* \in \mathcal{S}$ so that $z^* \equiv 0$ in (3.3). We fix $\mathcal{A}, \mathcal{B}$, and $K(\cdot)$ corresponding to $\mathbf{u}^*$ and note that (3.3) now gives

$$u' = -v^*z = \begin{cases} 0 & \text{on } \mathcal{A}, \\ -v^*z & \text{on } \mathcal{B}, \end{cases} \qquad \overline{v}' = \overline{u^*}z = \begin{cases} \overline{u^*}z & \text{on } \mathcal{A}, \\ 0 & \text{on } \mathcal{B}, \end{cases} \tag{6.2}$$

$$e^{\gamma\tau}z = \int_{-\infty}^{\tau} e^{\gamma\tilde{\tau}} \left\{ \begin{matrix} u^*\overline{v} & \text{on } \mathcal{A} \\ \overline{v^*}u & \text{on } \mathcal{B} \end{matrix} \right\} d\tilde{\tau}.$$

Now introduce

$$K^2 w := \begin{cases} -e^{\gamma\tau}u^*\overline{v} & (\text{so } v = -e^{-\gamma\tau}u^*\overline{w}) & \text{on } \mathcal{A}, \\ e^{\gamma\tau}\overline{v^*}u & (\text{so } u = e^{-\gamma\tau}\overline{v^*}w) & \text{on } \mathcal{B} \end{cases} \tag{6.3}$$

with w itself irrelevant, where $K = 0$; e.g., we may set $w := 0$ there. Then, noting that $\mathbf{u}^*$ is independent of ξ with $|v^*| = K$ on $\mathcal{A}$ and $|u^*| = K$ on $\mathcal{B}$, (6.2) gives (where $K \neq 0$)

[3] Any τ for which $K(\tau) = 0$ so $u = v = 0$ may arbitrarily be assigned either to $\mathcal{A}$ or to $\mathcal{B}$; the partition is unique to within $d\sigma$-nullsets.

$$w' = \frac{1}{K^2}\begin{Bmatrix} (-e^{\gamma\tau}u^*)(\overline{u^*z}) & \text{on } \mathcal{A} \\ (e^{\gamma\tau}\overline{v^*})(-v^*z) & \text{on } \mathcal{B} \end{Bmatrix} \tag{6.4}$$

$$= -e^{\gamma\tau}z = \int_{-\infty}^{\tau}\begin{Bmatrix} -e^{\gamma\tau}u^*\overline{v} & \text{on } \mathcal{A} \\ -e^{\gamma\tau}\overline{v^*}u & \text{on } \mathcal{B} \end{Bmatrix} d\tilde{\tau}.$$

If we define $\chi : \mathbb{R} \to \{\pm 1\}$ by

$$\chi := \{+1 \text{ on } \mathcal{A};\ -1 \text{ on } \mathcal{B}\} \tag{6.5}$$

and change variables to think of w as a function of (ξ, σ), using (2.1) so $K^2 d\tilde{\tau} = d\tilde{\sigma}$, then (6.4) takes the simple form

$$w' = \int_0^{\sigma} \chi w \, d\tilde{\sigma} \tag{6.6}$$

or, equivalently,

$$w_{\xi\sigma} = \chi w \quad \left(= \left[-e^{-\gamma\tau}z\right]_{\sigma}\right) \quad w(0,\cdot) = w_0 \in L^1_{d\sigma}. \tag{6.7}$$

Note that this formulation omits the irrelevant values of w for the set of τ where $K(\tau) = 0$, which disappears when we write things in terms of σ. Since the operator: $w \mapsto \int_0^{\sigma} \chi w \, d\tilde{\sigma}$ is certainly bounded, the solution operator $\mathbf{S}_{\xi} : w_0 \mapsto w(\xi, \cdot)$ for (6.6) forms a group on L^1. Alternatively, we might note that Theorem 3.4 ensures integrability of $K^2 w$ with respect to τ and so integrability of w with respect to σ.

We now restate the results of this discussion as a lemma without further proof.

LEMMA 6.1. *Let $\mathbf{u}^* \equiv \mathbf{u}_0^*$ be given in $\mathcal{S}$, determining $K(\cdot)$, $\sigma(\cdot)$ as in* (2.1) *and χ as in* (6.5)*; let $\mathbf{u}$ be a linearized perturbation, obtained from* (3.3)*, corresponding to a perturbation $\mathbf{u}_0 \in \mathcal{H}$ of the initial conditions $\mathbf{u}_0^*$. Then,*

$$u = \begin{cases} u_0, \\ e^{-\gamma\tau}v^* w, \end{cases} \qquad v = \begin{cases} -e^{-\gamma\tau}u^*\overline{w} & \text{on } \mathcal{A}, \\ v_0 & \text{on } \mathcal{B}, \end{cases} \tag{6.8}$$

where w, viewed as a function of (ξ, σ), satisfies (6.7) *and has initial data w_0 at $\xi = 0$ given by*

$$K^2 w_0 := \begin{cases} -e^{\gamma\tau}u^*\overline{v_0} & \text{on } \mathcal{A}, \\ e^{\gamma\tau}\overline{v^*}u_0 & \text{on } \mathcal{B}. \end{cases}$$

Note that w_0 is necessarily integrable with respect to σ.

LEMMA 6.2. *When χ is constant ($\chi \equiv \pm 1$), the solution of* (6.7) *is given explicitly by*

$$w(\xi, \sigma) := w_0(\sigma) + \xi \int_0^{\sigma} w_0(\tilde{\sigma})\Psi'(r) \, d\tilde{\sigma}, \tag{6.9}$$

where we set $r := \xi[\sigma - \tilde{\sigma}]$ and have

$$\Psi(r) := \begin{cases} J_0(2\sqrt{r}) & \text{for } \chi \equiv -1 \ (\mathbf{u}^* \in \mathcal{S}_0), \\ I_0(2\sqrt{r}) & \text{for } \chi \equiv +1, \end{cases} \tag{6.10}$$

where J_0 is the usual Bessel function of first kind and I_0 is the modified Bessel function: $I_0(s) := J_0(is)$.

Proof. We begin with the *ansatz* that $e^{-\gamma\tau}z$ can be obtained from w_0 by a convolution with respect to σ:

$$-e^{-\gamma\tau}z = \chi w_0 * \Psi(\xi\cdot) := \chi \int_0^\sigma w_0(\tilde{\sigma})\Psi(r)\,d\tilde{\sigma}$$

with $r := \xi[\sigma - \tilde{\sigma}]$. Differentiating with respect to σ then gives[4] (6.9)—provided we require $\Psi(0) = 1$ so as to have the correct condition at $\xi = 0$. Next, differentiating first with respect to ξ and then with respect to σ gives

$$\begin{aligned} w_\xi &= \int_0^\sigma w_0(\tilde{\sigma})\Psi'(r)\,d\tilde{\sigma} + \int_0^\sigma w_0(\tilde{\sigma})[r\Psi''(r)]\,d\tilde{\sigma} \\ &= \int_0^\sigma w_0(\tilde{\sigma})[r\Psi'(r)]'\,d\tilde{\sigma}, \\ w_{\xi\sigma} &= w_0(\sigma)[r\Psi'(r)]'\,|_{r=0} + \xi \int_0^\sigma w_0(\tilde{\sigma})[r\Psi'(r)]''\,d\tilde{\sigma} \\ &= \chi\left[w_0(\sigma) + \xi\int_0^\sigma w_0(\tilde{\sigma})\Psi'(r)\,d\tilde{\sigma}\right] \end{aligned}$$

with the final equality coming from (6.7). We obtain this last, provided that

$$[r\Psi'(r)]'\,|_{r=0} = \chi, \qquad [r\Psi'(r)]'' = \chi\Psi'(r).$$

The differential equation gives $([r\Psi'(r)]' - \chi\Psi(r)) =$ constant and evaluating at $r = 0$ shows this constant must be zero. If we now set $\Psi(r) =: \Phi(s)$ with $s := 2\sqrt{r}$, this gives

$$s^2\Phi'' + s\Phi' - \chi s^2\Phi = 0, \qquad \Phi(0) = 1.$$

For $\chi = -1$, this is Bessel's equation with parameter $a = 0$ and the normalization gives $\Phi(s) = J_0(s)$; for $\chi = +1$, we then get $\Phi(s) = J_0(is) =: I_0(s)$.

Note that $d\Psi(r)/dr = \Phi'(2\sqrt{r})/\sqrt{r}$ so, since

$$J_0'(s) = -J_1(s) \qquad I_0'(s) = iJ_0'(is) = -iJ_1(is) =: I_1(s),$$

we have

$$\Psi'(r) = \begin{cases} -J_1(2\sqrt{r})/\sqrt{r} & \text{for } \chi = -1, \\ I_1(2\sqrt{r})/\sqrt{r} & \text{for } \chi = +1 \end{cases} \tag{6.11}$$

for use in (6.9). Since $J_0(z)$ is an *even* analytic function of z, it is also analytic in $\sqrt{z}$ so Ψ is analytic (a fortiori bounded) near zero and so on any bounded interval. Thus, (6.9) makes sense for all $d\sigma$-integrable w_0. □

Remark 6.3. What information can we draw from this in the case of $\mathbf{u}^* \in \mathcal{S}_0$ (so $\chi \equiv -1$)? We observe, first, that the integral term in (6.9) is a convolution so, using L^1 norms,

$$\|w(\xi,\cdot)\| \leq \|w_0\|\,[1 + \|\xi\Psi'\|]\,.$$

[4] It is at this point already that we need the assumption: $\chi \equiv$ constant.

Here we have, setting $s^2 = \xi\sigma$,

$$\|\xi\Psi'\| := \int_0^{\kappa^2} |\Psi'(\xi\sigma)|\,\xi d\sigma = 2\int_0^{\kappa\sqrt{\xi}} |J_1(2s)|\,ds,$$

and since $J_1(s)$ decays like $1/\sqrt{s}$, we see that we have $\|\xi\Psi'\| = \mathcal{O}(\xi^{1/4})$ so, at worst, we always have

$$\|w(\xi,\cdot)\|_{L^1} = \mathcal{O}(\xi^{1/4}) \quad \text{as } \xi \to \infty. \tag{6.12}$$

On the other hand, if we are considering perturbations for which $w_0 \in BV$ (bounded variation) so that it is justifiable to integrate by parts in (6.9), then we obtain

$$w(\xi,\sigma) = w_0(0)J_0(2\sqrt{\xi\sigma}) + \int_0^{\sigma} J_0(2\sqrt{\xi(\sigma-\tilde{\sigma})})\,dw_0(\tilde{\sigma}). \tag{6.13}$$

The integral term on the right can be estimated in the same way as for (6.12) to give $\mathcal{O}(\xi^{-1/4})$ decay. The first term goes to zero pointwise in σ as $\xi \to 0$ at a rate $\mathcal{O}(\xi^{-1/4})$. This is not uniform, but can certainly be integrated in σ to give a decay rate

$$\|w(\xi,\cdot)\|_{L^1} = \mathcal{O}(\xi^{-1/4}) \quad \text{as } \xi \to \infty \tag{6.14}$$

in this case.

Clearly, even as "linearized stability," this is far weaker than the results we have for the discretized setting, corresponding to Lemma 5.5 and Theorem 5.6. Nevertheless, it is as strong a result as we have been able to obtain here. That the result is weaker is certainly related to the fact that $\mathcal{S}_0$ is not isolated from other stationary solutions, but we might still hope to improve this. Indeed, if (6.12) could be improved to give boundedness as $\xi \to \infty$ for each initial $w_0 \in L^1(0,\kappa^2)$, then a simple argument would show decay for all w_0. So far we do not know whether this is true and, further, note that this linearized stability by itself would not show $\mathbf{u} \to \mathcal{S}_0$ (locally) for the nonlinear problem.

Remark 6.4. We next consider the case of $\mathbf{u}^* \in \mathcal{S}\backslash\mathcal{S}_0$ so $\mathcal{A}$ is nonempty. We are seeking here to demonstrate instability, so it is only necessary to construct special examples. Suppose $\mathcal{A}$ contains an interval $[\sigma_-,\sigma_+]$ and we take the perturbation $\mathbf{u}_0$ so $w_0 \equiv 1$ on this interval and vanishes otherwise. This effectively lets us take $\sigma_- = 0$ with no loss of generality. Thus, at least for σ in the interval, we are considering (6.7) with $\chi \equiv +1$ and Lemma 6.2 applies. Integrating by parts as for (6.13), we then have

$$w(\xi,\sigma) = I_0(2\sqrt{\xi\sigma}) \quad \text{for } 0 < \sigma < \sigma_+$$

and, since I_0 grows exponentially, we have instability: growth of $w(\xi,\cdot)$ which is exponential in $\xi^{1/2}$. We remark that this is consistent with having $\|w\| = \mathcal{O}(e^{\varepsilon\xi})$ for arbitrarily small $\varepsilon > 0$, as is suggested by the fact that the Volterra operator on the right of (6.6) has spectrum $\{0\}$.

What does this tell us for the nonlinear problem? Using the fact that, as observed in Remark 4.3, the solution map has a locally bounded second derivative we can show that *for arbitrarily large M and arbitrarily small $\varepsilon > 0$ there exist solutions of* (2.5) *which initially differ from $\mathbf{u}^*$ by less than ε but at a later time differ by more than $M\varepsilon$, assuming $M\varepsilon$ is not too big.*

Again, as in Remark 6.3, this is not very much. It certainly is enough, however, to guarantee that the exponential factor in (3.2) cannot be omitted to give the Lipschitz continuity uniform in ξ.

7. Asymptotic behavior. In this section we consider the asymptotic behavior of solutions of the system (2.5) as $\xi \to \infty$. We will assume that the reduction of §4 has been made, if necessary, so in (2.1) we have $K(\tau) \equiv 1$ for $\tau \in [0,1]$ with everything vanishing for $\tau \notin [0,1]$ whence we have (4.5) and (4.4). While we might conjecture for this context essentially the same results which we obtained for the discretized system in the previous section, we have so far not been able to carry out this program completely. Our results here are primarily the consequences of (4.1).

Remark 7.1.

DEFINITION. A function $\mathbf{v} : \mathbb{R}_+ \to \mathcal{X}$ will be called **recurrent** (for some ξ_0) if there is a sequence $\xi_n \to \infty$ for which $\|\mathbf{v}(\xi_n) - \mathbf{v}(\xi_0)\| \to 0$.

Clearly, every periodic function is recurrent for arbitrary ξ_0. Following Bohr, $\mathbf{v}$ is *almost periodic* on $\mathbb{R}_+$ if, for any $\varepsilon > 0$, there exists $\ell(\varepsilon)$ such that (with $\tilde{\xi}$ arbitrary) $\|\mathbf{v}(\xi) - \mathbf{v}(\xi_\varepsilon)\| \le \varepsilon$ for some $\tilde{\xi} < \xi_\varepsilon < \tilde{\xi} + \ell(\varepsilon)$ and all $\xi \in \mathbb{R}_+$; clearly, this also implies recurrence for arbitrary ξ_0. This would include sums of (incommensurately) periodic functions: if $\mathbf{v} := \sum_j \mathbf{v}_j$ (where each $\mathbf{v}_j$ is continuous with period ξ_j and $\sum_j \sup_\xi |\mathbf{v}_j|$ convergent); then it is easily seen to be almost periodic, using the number-theoretic result that we can always find positive integers q and $\{n_j\}$, making $|q - n_j \xi_j|$ arbitrarily small simultaneously for $j = 1, \cdots, J$ [6, Thm. 201] every "positive ray" $q[1/\xi_1, \cdots, 1/\xi_J]$ passes arbitrarily close to integer lattice points in $\mathbb{R}_+^J$ for infinitely many integers q].

Finally, we note that, for $\mathbf{v}$ satisfying an autononomous ODE, if we were to have $\|\mathbf{v}(\xi_n) - \mathbf{v}(\xi_0)\| \to 0$ for any sequence $\{\xi_n\}$ bounded away from ξ_0, then we would have recurrence at ξ_0. To see this when $\{\xi_n\}$ is bounded, extract a subsequence converging to some $\xi_1 \ne \xi_0$ and observe that continuity gives $\mathbf{v}(\xi_1) = \mathbf{v}(\xi_0)$ whence, assuming uniqueness for the ODE, $\mathbf{v}$ would necessarily be periodic with period $|\xi_1 - \xi_0|$.

THEOREM 7.2. *Under the hypotheses of Lemma* 4.1 *and taking the system reduced as in the previous section, if* $\mathbf{u}$ *is recurrent (for some* ξ_0*), then it is stationary:* $z \equiv 0$ *so* $\mathbf{u}$ *independent of* ξ.

Proof. By (4.1.ii), for each τ and for $\xi_n - \xi_0 \ge \delta > 0$ we have

$$\int_{\xi_0}^{\xi_0+\delta} |z(\cdot,\tau)|^2 \le \int_0^1 |v(\xi_n,\cdot)|^2 - \int_0^1 |v(\xi_0,\cdot)|^2 \le \sqrt{2\tau}\|\mathbf{u}(\xi_n,\cdot) - \mathbf{u}(\xi_0,\cdot)\| \to 0,$$

which gives $z \equiv 0$ on $[\xi_0, \xi_0 + \delta] \times [0,1]$ — and so everywhere, as in Corollary 4.2, giving stationarity of $\mathbf{u}$ as asserted. □

THEOREM 7.3. *Under the hypotheses of Lemma* 4.1 *and taking the system reduced as in the previous section, we always have* $z \in L^2(\mathbb{R}_+ \times [0,1])$ *and uniform convergence:* $z(\xi,\cdot) \to 0$ *as* $\xi \to \infty$.

Proof. It is convenient to set

$$U(\cdot,\tau) := \int_0^\tau |u|^2, \qquad V(\cdot,\tau) := \int_0^\tau |v|^2. \tag{7.1}$$

From (4.4) we see that $0 \le U, V \le \tau$ and, also using (4.1), we see that each is uniformly Lipschitzian (jointly in ξ, τ) with $U' = -|z|^2$, $V' = |z|^2$ so U is nonincreasing in ξ and V nondecreasing. For each (fixed) τ, we have

$$\begin{aligned}\int_\xi^{\tilde{\xi}} |z(\cdot,\tau)|^2 &= U(\xi,\tau) - U(\tilde{\xi},\tau) \le U(\xi,\tau) \le \tau \\ &= V(\tilde{\xi},\tau) - V(\xi,\tau) \le V(\tilde{\xi},\tau) \le \tau.\end{aligned}$$

So, taking $\xi = 0$ and letting $\tilde{\xi} \to \infty$, we see that we have $\|z(\cdot,\tau)\| \le \sqrt{\tau}$ (this is the $L^2(\mathbb{R}_+)$-norm) whence $\|z\| \le 1/\sqrt{2}$. (Here, this is the $L^2(\mathbb{R}_+ \times [0,1])$-norm.) Using the bounds $0 \le |u|^2, |v|^2 \le 1$ in (4.1.iii), we have $|z'| \le \sqrt{\tau}\|z\|$ so $\|z'\| \le \|z\|/\sqrt{2}$; thus

$$\left|\left(\|z\|^2\right)'\right| = 2\left|\mathrm{Re}\int_0^1 \bar{z}z'\right| \le \sqrt{2}\,\|z\|^2.$$

Then $(\|z\|^2)'$ is integrable on $\mathbb{R}_+$ whence $\|z\|^2$ has a limit as $\xi \to \infty$, necessarily zero almost everywhere in τ since $z \in L^2(\mathbb{R}_+ \times [0,1])$. This gives $z(\xi,\cdot) \to 0$ in $L^2(0,1)$-norm. Since $z(\xi,\cdot)$ stays in a compact subset of $C[0,1]$, as we have noted in the previous section, this is actually uniform convergence. □

Remark 7.4. Since the set $\mathcal{S}$ of stationary solutions is characterized by having $z \equiv 0$, it would be tempting to conclude, from the uniform convergence above, that it is a global attractor, i.e., that we must necessarily have $\mathbf{u} \to \mathcal{S}$; indeed, computation suggests the stronger conjecture that we have $\mathbf{u} \to \mathcal{S}_0$ $(u \to 0)$ as $\xi \to \infty$ for all $\mathbf{u}_0 \in \mathcal{U}_1$ — except, of course, $\mathcal{S}\backslash\mathcal{S}_0$. We do note, however, that this cannot follow simply from Theorem 7.3 since, e.g., there exist sequences like $\mathbf{u}_j(\tau) := (\cos j\pi\tau, \sin j\pi\tau)$ for which we have $|z_j(\tau)| \le 1/4\pi j \to 0$, but $\|\mathbf{u}_j - \mathcal{S}\| = 2\sqrt{\pi - 2} \not\to 0$. (Of course, this example has *not* been constructed by taking $\mathbf{u}_j := \mathbf{u}(\xi_j)$ with $\xi_j \to \infty$ for a solution $\mathbf{u}$ of (2.5).)

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