Non-Gaussian corrections to the Gordon–Haus distribution resulting from soliton interactions

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Received September 12, 1994

In a soliton transmission system, spontaneous emission noise owing to optical amplifiers leads to timing jitter that is usually assumed to be Gaussian distributed. It is shown that the mutual interaction of solitons in neighboring time slots can lead to non-Gaussian tails on the distribution function and to a substantial increase in the bit-error rate. It is argued that the approach used here will also be of use in the study of non-return-to-zero systems.

In modern-day optical fiber communication systems, one aspires to achieve bit-error rates that are lower than 10^{-12}. Theoretical calculations to date, both analytical and computational, of the bit-error rate rely on the suspect assumption that the quantities of interest, for example, the temporal location of solitons in a soliton system, will be Gaussian distributed in the presence of noise. With this assumption, one merely needs to know the standard deviation of the quantity of interest to determine the bit-error rate.

In this Letter I will calculate the probability distribution function for the soliton temporal location in a filtered soliton system, and I will show that the tails of the distribution function are non-Gaussian, substantially affecting the bit-error rate without significantly changing the standard deviation. The system studied here is of current experimental interest[1]; however, the basic purpose of this Letter is not to explain the details of the experiments but to illustrate with a simple yet realistic example the importance of non-Gaussian tails and to show how to calculate the actual distribution function. For this reason I will take into account only the contribution of spontaneous emission that leads to Gordon–Haus jitter,[2] leaving aside for the present the contributions that are due to acoustic and polarization effects, although these are quite important in the experiments.[1]

The basic physical mechanism that leads to non-Gaussian tails is the mutual interaction between solitons in neighboring time slots.[3] This interaction is typically small, but when by chance two solitons in neighboring time slots are displaced toward each other by roughly half the temporal distance to the time-slot boundary, then the mutual interaction becomes significant and the solitons are pulled closer to the time-slot boundary. Schematically the probability that a single isolated soliton is displaced by a time q_{GH} from the center of the slot as a result of the Gordon–Haus effect is proportional to $\exp(-q_{\mathrm{GH}}^2/2\sigma_{\mathrm{GH}}^2)$; I will give below an explicit expression for the Gordon–Haus standard deviation σ_{GH}. It follows that the probability that two solitons in neighboring time slots are both displaced by an amount $\pm q_{\mathrm{GH}}$ in the absence of the mutual interaction is proportional to $\exp(-q_{\mathrm{GH}}^2/\sigma_{\mathrm{GH}}^2)$, which is considerably smaller than $\exp(-q_{\mathrm{GH}}^2/2\sigma_{\mathrm{GH}}^2)$ when $q_{\mathrm{GH}}/\sigma_{\mathrm{GH}} \gg 1$. However, if both solitons are actually displaced by an amount $\pm q_M$ owing to the mutual interaction, where $q_M > \sqrt{2} q_{\mathrm{GH}}$, then the probability distribution function will be enhanced at the temporal offset $q = \pm q_M$.

From a fundamental viewpoint, a single isolated soliton is a natural mode of the optical fiber[4]; thus its response to amplified spontaneous emission noise will be essentially linear, and its temporal offset will be Gaussian distributed. However, interacting solitons are not individually pure modes, and it is here that the nonlinearity of the optical fiber transmission channel makes itself felt.

The stochastic differential equations that describe two filtered solitons with the same phase in neighboring time slots may be written as[5]

$$\begin{aligned} \ddot{q}_1 &= -\gamma \dot{q}_1 + \alpha S_1 + 4\exp[-(q_2 - q_1)], \\ \ddot{q}_2 &= -\gamma \dot{q}_2 + \alpha S_2 - 4\exp[-(q_2 - q_1)], \end{aligned} \qquad (1)$$

where I am using normalized soliton units and assuming that the solitons have unity amplitude. The overdots indicate derivatives with respect to z, the normalized distance along the fiber. The quantity γ gives the effect of the filter, S_1 and S_2 are Gaussian-distributed independent white-noise sources with zero mean and unity variance, and α is the noise figure. The expression that I use for the mutual interaction was first derived by Gordon.[3] In the experiments to date, the solitons are highly phase coherent when injected into the fiber and remain phase coherent for the lengths of the experiments.[6] The phase difference between neighboring solitons can be chosen by the experimentalist, with zero phase difference being the worst case since the mutual interaction is largest. Moreover, the sliding filters that were used in recent experiments[1] lead to a constantly rotating phase difference that tends to reduce the mutual interaction. This effect is an important factor in the success of sliding filters, but it seriously complicates the calculations. In the interest of keeping the discussion simple and illustrative, I therefore neglect it.

The approach that I use to determine the probability distribution function is a variant of the method of characteristics or the path-integral method.[7] The

0146-9592/95/030285-03$6.00/0

random process generated by Eqs. (1) is a Markov process in the variables $\mathbf{Q} = (q_1, \dot{q}_1, q_2, \dot{q}_2)$. Indeed, for distances $\gamma z \gg 1$, the reduced process $\mathbf{q}(z) = [q_1(z), q_2(z)]$ is a Markov process, which is a significant simplification. This reduction is not possible in the unfiltered system; thus the filtered system is actually easier to analyze. One can now in principle proceed as follows: In the absence of the mutual interaction, the probability distribution function $f_{\text{GH}}(\mathbf{q})$ is known. There are different paths over which $\mathbf{q} \to \mathbf{q}_{\text{GH}}$ as $z \to z_f$, where z_f is the final z value, corresponding to different realizations of S_1 and S_2. When the mutual interaction is present, the modified $\mathbf{q}$ value $\mathbf{q}_M(\mathbf{q}_{\text{GH}}, \mathcal{P})$ does not depend uniquely on $\mathbf{q}_{\text{GH}}$ but also on the path $\mathcal{P}$ by which $\mathbf{q}_{\text{GH}}$ is reached. The probability distribution function $f_M(\mathbf{q}_M)$ is given by

$$f_M(\mathbf{q}_M) = \int \mathrm{d}\mathbf{q}_{\text{GH}} \int \mathrm{d}\mathcal{P} f_{\text{GH}}(\mathbf{q}_{\text{GH}}) \times \delta[\mathbf{q}_M - \mathbf{q}_M(\mathbf{q}_{\text{GH}}, \mathcal{P})], \qquad (2)$$

where the integral $\int \mathrm{d}\mathcal{P} \ldots$ implies an appropriately weighted integral over all possible paths.[7] A discretization of Eq. (2) leads to a computationally intensive yet tractable and accurate calculation of the distribution function and ultimately of the bit-error rate. A complete discussion of this issue is left for the future.

Here instead I use a maximum-likelihood approach that is less accurate but reduces the problem to one that can be solved in a few minutes on a workstation. There is one path $\mathcal{P}_{\max}$ leading to $\mathbf{q}_{\text{GH}}$ that is most likely to occur. Since any path that is reasonably likely to occur will be fairly close to this path, it is reasonable to eliminate the integral $\int \mathrm{d}\mathcal{P} \ldots$ in Eq. (2) by using only $\mathcal{P}_{\max}$ with each $\mathbf{q}_{\text{GH}}$, defining a unique relationship $\mathbf{q}_M(\mathbf{q}_{\text{GH}}) = \mathbf{q}_M(\mathbf{q}_{\text{GH}}, \mathcal{P}_{\max})$. The most likely way to achieve a separation $q_{\text{sep}} = q_2 - q_1$ is for each soliton to be displaced by an equal amount in opposite directions, so that Eqs. (1) become

$$\ddot{q} = -\gamma \dot{q} + \alpha S_{\max} + 4 \exp[-2(q_l - q)], \qquad (3)$$

where q is the displacement of either q_1 and q_2 from the center of the time slot, q_l is the time-slot boundary, and $S_{\max}$ is the contribution of the noise source along $\mathcal{P}_{\max}$. When the displacement of q including the effect of the mutual interaction, q_M, is less than $\sqrt{2}$ times the displacement of q excluding the effect of the mutual interaction, q_{GH}, then the contribution to the distribution function $f_M(q_M)$ is dominated by $f_{\text{GH}}(q_M)$. By contrast, when $q_M > \sqrt{2} q_{\text{GH}}$, then the contribution to the distribution function is dominated by $f_{\text{GH}}{}^2[q_{\text{GH}}(q_M)]$, where $q_{\text{GH}}(q_M)$ is obtained by inverting $q_M(q_{\text{GH}}, \mathcal{P}_{\max})$. Thus one may approximate $f_M(q_M)$ by the expression

$$f_M(q_M) \simeq \frac{\frac{1}{2} f_{\text{GH}}(q_M) + \frac{1}{2} f_{\text{GH}}{}^2[q_{\text{GH}}(q_M)]}{\int_{-\infty}^{\infty} \mathrm{d}q_M' \left\{ \frac{1}{2} f_{\text{GH}}(q_M') + \frac{1}{2} f_{\text{GH}}{}^2[q_{\text{GH}}(q_M')] \right\}}. \qquad (4)$$

To determine $\mathcal{P}_{\max}$, I first note that the probability distribution function in the absence of the mutual interactions is given by

$$f_{\text{GH}}(q_{\text{GH}}) = \frac{\gamma}{\alpha \sqrt{2\pi z}} \exp(-\gamma^2 q_{\text{GH}}{}^2 / 2\alpha^2 z), \qquad (5)$$

when $\gamma z \gg 1$. Since the random process that generates f_{GH} is a Markov process, it follows that

$$f_{\text{GH}}(q, z; q_f, z_f) = \frac{\gamma^2}{2\pi\alpha^2 z^{1/2} (z_f - z)^{1/2}} \times \exp\left\{ -\frac{\gamma^2}{2\alpha^2} \left[\frac{q^2}{z} + \frac{(q_f - q)^2}{z_f - z} \right] \right\}, \qquad (6)$$

where $f_{\text{GH}}(q, z; q_f, z_f)$ is the joint probability distribution function for the soliton's arriving at time q after a distance z and then arriving at time q_f after a distance z_f. Completing the square for q, one finds that

$$f_{\text{GH}}(q, z; q_f, z_f) = \frac{\gamma^2}{2\pi\alpha^2 z^{1/2} (z_f - z)^{1/2}} \times \exp\left[-\frac{\gamma^2}{2\alpha^2} \frac{z_f}{z(z_f - z)} \left(q - \frac{z}{z_f} q_f \right)^2 - \frac{\gamma^2}{2\alpha^2 z_f} q_f{}^2 \right]. \qquad (7)$$

Hence one finds that $q = z q_f / z_f$ along $\mathcal{P}_{\max}$, which implies that $\dot{q} = q_f / z_f$ and $\ddot{q} = 0$ along $\mathcal{P}_{\max}$. Using Eq. (3) and noting that, in the absence of the mutual interaction, $\ddot{q} = -\gamma \dot{q} + \alpha S_{\max}$, I conclude that $S_{\max} = \gamma q_f / \alpha z_f$. Using this value of $S_{\max}$ in Eq. (3) permits one to compute $q_M(q_{\text{GH}}, \mathcal{P}_{\max})$.

I now consider a specific set of parameters, $\gamma = 0.4$, $q_l = 5.5$, and $z_f = 250$, that correspond closely to current experimental values at 10 Gbits/s over 35 Mm.[1] I also set $\alpha = 10^{-2}$, which is a bit higher than the theoretically predicted value, but I neglect the acoustic and polarization effects that enhance the jitter. Figure 1 shows the solution of Eq. (3) for $q_M(z)$ at special values of $q_f = q_{\text{GH}}(z_f)$. The function $q_M(z)$ grows nearly linearly in all cases until $q_M \simeq 2.5$, at which point it increases rapidly. In the range $q_f = 0$–1, $q_M(z_f)$ hardly differs from q_f, but beyond $q_f = 1$ the deviations become significant. Figure 2 shows $q_{\text{GH}}(q_M)$, which is obtained by inverting $q_M(q_{\text{GH}}, \mathcal{P}_{\max})$ at $z = z_f$. Beyond $q_M \simeq 2.5$ the curve flattens, and

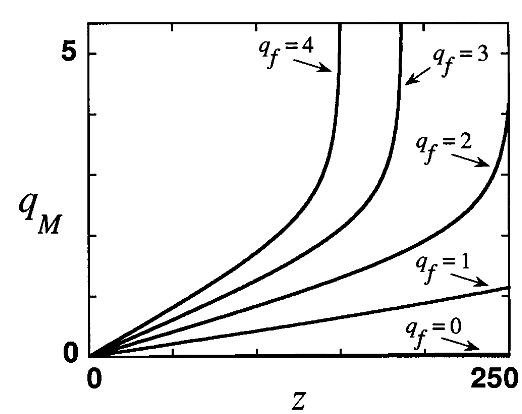

Fig. 1. Solution for $q_M(z)$ determined from Eq. (3) at some special values of $q_f \equiv q_{\text{GH}}(z_f)$.

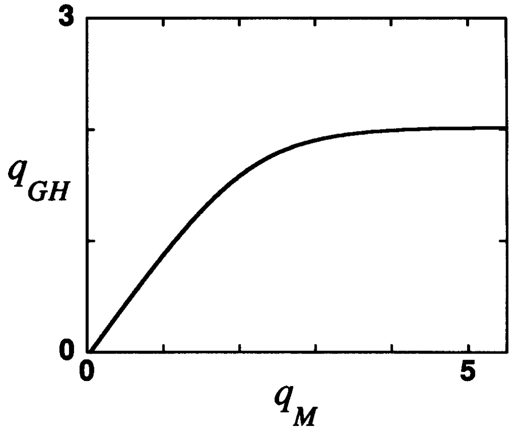

Fig. 2. Solution for $q_{\mathrm{GH}}(z_f)$ as a function of $q_M(z_f)$. I calculated this function by first finding $q_M(q_f)$ at z_f and then inverting, after recalling that $q_{\mathrm{GH}}(z_f) = q_f$.

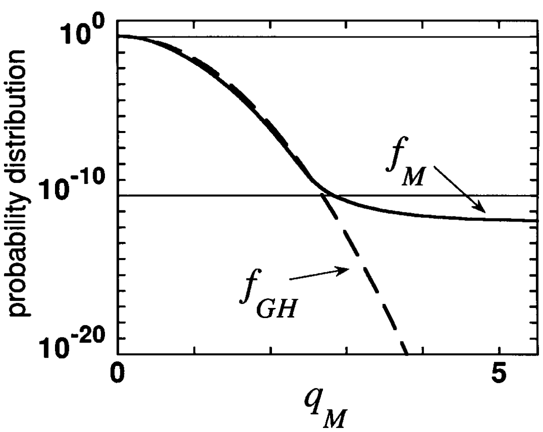

Fig. 3. Comparison of the probability distribution functions f_{GH} and f_M. A significant increase in f_M relative to f_{GH} is visible beyond $q_M = 2.5$.

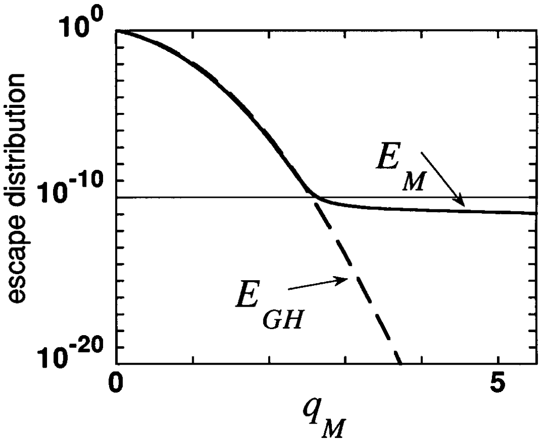

Fig. 4. Comparison of the escape distribution functions E_{GH} and E_M. These functions are directly related to the bit-error rate.

one finds that $q_{\mathrm{GH}} = q_M/\sqrt{2}$ when $q_M = 2.55$. Beyond this value a large enhancement of the probability distribution function is expected. Using relation (4) to plot $f_M(q_M)$, one finds that a significant tail is visible beyond $q_M \simeq 2.5$, as shown in Fig. 3. I next define the escape distribution function:

$$E_M(q_M) = 2\int_{q_M}^{\infty} f_M(q_M{}')\mathrm{d}q_M{}', \qquad (8)$$

which gives the probability that $|q| > q_M$ and is shown in Fig. 4. Assuming that there is an error when $|q| > q_{\mathrm{error}}$, $E_M(q_{\mathrm{error}})$ equals the bit-error rate. For the parameters of Ref. 1, $q_{\mathrm{error}} = 4.5$. In the example considered here, $E_M(4.5)$ is in excess of 10^{-12}, which is measurable, whereas the assumption that the probability distribution function is Gaussian yields a number that is too small to measure.

In summary, I have shown that the mutual interaction of solitons in neighboring time slots can lead to non-Gaussian tails in the distribution of the soliton jitter and to a substantial increase in the bit-error rate. Although the physical mechanism being considered here is specific to solitons, the basic physical idea is not. Optical fibers are nonlinear channels, and there is no reason that Gaussian noise should lead to a Gaussian distribution of the quantities of interest. Indeed, one intuitively expects the deviations from a Gaussian distribution to become important for those rare events in which the noise-induced changes in the physical quantities of interest are large—just the sorts of events that are likely to cause errors. Thus I anticipate that non-Gaussian tails are of importance in non-return-to-zero systems as well as in soliton systems.

It is a pleasure to acknowledge the strong encouragement that I received to study this problem from M. Kruger, L. Mollenauer, and P. Runge. I am grateful to E. Golovchenko, A. Pilipetskii, L. Mollenauer, and P. Mamyshev for useful comments on this manuscript. This study was supported by the Advanced Research Projects Agency through the U.S. Air Force Office of Scientific Research and by the National Science Foundation.

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