A Covariance Matrix Method to Compute Bit Error Rates in a Highly Nonlinear Dispersion-Managed Soliton System

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***Abstract*— We completely describe the covariance matrix method for the first time, and we use it to compute the noise evolution in a 10 Gb/s single-channel dispersion-managed soliton system propagating over 24,000 km. The linearization assumption upon which the covariance matrix method is based breaks down, unless we explicitly separate the phase and timing jitter of each pulse from the noise. We describe a procedure for carrying out this separation.**

***Index Terms*— Optical fiber communication, Amplifier noise, Phase jitter, Optical Kerr effect, Linear approximation, Monte Carlo methods, Receivers, Spectral analysis.**

I. Introduction

AMPLIFIED spontaneous emission (ASE) noise that optical amplifiers add to the signal gives rise to bit errors and sets the lower limit on the signal power. One traditional way of computing bit error rates (BERs) and eye diagrams is to run Monte Carlo simulations and extrapolate the results under the assumption that the electrical power at the receiver after narrow-band filtering is Gaussian-distributed in the marks (Ones) and spaces (Zeros). This method leads to large statistical fluctuations in the tails of the probability density function (pdf) and is hence inefficient. As a consequence, system designers often use a simplified approach in which they assume that the optical noise spectrum at the receiver is white, effectively neglecting the nonlinear signal-noise interaction. This simplification is often inappropriate for long-haul optical communications systems. In this letter, we report on the application of the covariance matrix method to calculate the pdfs of the received voltage in a 10 Gb/s single-channel dispersion-managed soliton (DMS) system with a transmission distance of 24,000 km [1]. This method is based on the linearization assumption that the noise does not interact with itself during propagation through the fiber in an appropriate basis set [2], [3]. The pulses in the DMS system considered here are periodically stationary and the signal propagation is highly nonlinear. Hence, this system poses a stringent test of our approach.

Previous study of the same system [2] showed that the linearization assumption breaks down unless we use a basis set for the covariance matrix in which phase and timing jitter are separated from the other noise components. This separation is necessary because the nonlinear equations that govern the fiber transmission imply that small amounts of amplitude and frequency noise can lead to large amounts of phase and timing jitter respectively, which in turn lead to a breakdown of the linearization assumption in the standard Fourier basis. By contrast, if we modify the basis to separate out the noise components whose first-order contribution generates phase and timing jitter, we find that the coefficients of the modified Fourier basis, along with the jitter, obey the linearization assumption and remain multivariate Gaussian distributed far longer than the original Fourier coefficients [2]. This result is similar to one that is well known in the theory of solitons, where it is standard to use a basis set that consists of discrete as well as continuous components, rather than the usual Fourier basis, when studying the effects of perturbations and noise [4].

In this letter, we show how to apply the linearization assumption to directly calculate the covariance matrix for a highly nonlinear dispersion-managed soliton system. This work extends previous work in which we used extensive Monte Carlo simulations to calculate the covariance matrix for the same DMS system that we describe here [2], except that the path average dispersion is lower in this simulation, leading to a lower timing jitter. The Monte Carlo approach to calculating the covariance matrix, as described in [2], serves to validate the linearization assumption, but it requires an order of magnitude more computational time than the direct approach described here and is inherently less accurate. This work also extends previous work in which we directly calculated the covariance matrix for a chirped return to zero system [3]. In this earlier work, it was not necessary to separate the timing jitter from the other noise modes, which significantly simplifies the algorithm. Thus, this work contains the first complete description of the covariance matrix method.

The basic approach that we use is to follow the evolution of the standard Fourier basis, projecting out the contribution to the phase and the central time shift from the Fourier coefficients at each amplifier. At the end of the transmission line, there is a square law receiver that is insensitive to the phase, so we do not

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need to keep track of the phase. We reintroduce the effect of the timing jitter by applying a convolution, as described in [2]. The phase and timing jitter could in principle be separated from the covariance matrix at the end of the transmission line. There is a loss of numerical accuracy, unless the jitter is incrementally separated.

Our approach is fully deterministic and does not rely at all on Monte Carlo simulations, which allows us to greatly increase the accuracy at small BERs at a fraction of the computational cost of Monte Carlo simulations. To validate our results, we compare them to standard Monte Carlo simulations. The nonlinearity in the DMS system is significantly higher than in the majority of all modern transmission systems, and hence we expect our approach of use in almost any system.

II. Simulation Procedure

The simulated 10 Gb/s transmission line models a recirculating loop and consists of 225 periods of a dispersion map 106.7 km long. A complete description of the system and the simulation is given in [1]. Each map contains a fiber span 100.24 km long and normal dispersion of -1.096 ps/nm-km and a span of length 6.71 km and anomalous dispersion of 16.696 ps/nm-km. The path average dispersion equals the experimental value of 0.02 ps/nm-km; we previously simulated a system with 0.08 ps/nm-km [2]. Third-order dispersion is not relevant in this system [1] and is set to zero. The carrier wavelength is 1551.49 nm, matching the experimental value. The fiber loss of 0.21 dB/km is compensated by five EDFAs. We assume an effective fiber area of 49 $(\mu\text{m})^2$ for both fiber types. One EDFA follows each of the four 25-km segments of normal-dispersion fiber, and the fifth follows the segment of anomalous-dispersion fiber. There is a 2.8 nm (350 GHz) optical bandpass filter in each map period to reduce the amount of noise.

We model the amplifiers as EDFAs with static gain, as opposed to explicitly including gain saturation. We carefully adjust the static gains so that they equal the effective gains one would obtain using EDFAs with a saturation time of 1 ms and a saturation power of 10 mW in accordance with [2]. The spontaneous emission factor is $n_{\text{sp}} = 1.2$. The launched pulses have a Gaussian shape with a FWHM duration of 9 ps and a peak power of $P_{\text{peak}} = 8$ mW. The peak pulse power at the beginning of the normal span is 3.1 mW. The signal is injected and received at the chirp-free point near the middle of the anomalous span. We transmit the 8-bit PRBS sequence 11101000 in a total simulation time window of $T = 800$ ps, which includes all possible three-pulse sequences. Since the solitons do not spread significantly during the transmission, there is no need to study longer sequences of marks and spaces. The average signal power is -4.2 dBm. We model the receiver as an ideal square law detector followed by an electrical low-pass 5th-order Bessel filter with a one-sided 3-dB bandwidth of 4.3 GHz.

To assess the degree of system nonlinearity, we define a nonlinear scale length L_{nl} as the length over which a nonlinear phase rotation of 2π occurs. We may write $L_{\text{nl}} = 1/(\gamma P_{\text{peak}})$ with the nonlinearity coefficient $\gamma = 2.1\,(\text{W}\cdot\text{km})^{-1}$. The transmission distance of 24,000 km is 400 times larger than L_{nl}, showing that the DMS system is highly nonlinear. By contrast, commercial systems are 3–5 times the nonlinear length scale at most [5].

We use the split-step Fourier method to solve the scalar nonlinear Schrödinger equation, which only takes into account one optical polarization. In the recirculating loop that we are modeling, the polarization dependent loss (PDL) is large and the polarization controllers are optimized to pass the signal with minimum loss. Consequently, the signal is dominated by one polarization, and the orthogonal polarization can be neglected.

We proceed by first expressing the optical field envelope as $u = u_0 + \delta u$, where $u_0 = \langle u \rangle$ is the noise-free field, and δu represents accumulated noise. We next write $\delta u = \sum_{k=-N_{\text{FFT}}/2}^{N_{\text{FFT}}/2-1} [\alpha_k + i\beta_k] \exp(-i\omega_k t)$, where α_k and β_k are N_{FFT} real and imaginary noise Fourier coefficients and $\omega_k = 2\pi k/T$.[1] We define the real vector $\boldsymbol{a} = (\alpha_{-N/2}, \alpha_{-N/2+1}, \ldots, \alpha_{N/2-1},\ \beta_{-N/2}, \beta_{-N/2+1}, \ldots,$ $\beta_{N/2-1})^t$ of length $2N$, where the symbol t denotes the transpose. We choose $N_{\text{FFT}} = 2048$ and $N = 120$ in this work.

The evolution of the $2N \times 2N$ noise covariance matrix $\mathcal{K} = \langle \boldsymbol{a}\boldsymbol{a}^t \rangle$ over one fiber leg from $z = 0$ to $z = L$, in which no noise is added, followed by an EDFA with gain G, is given by

$$\mathcal{K}(L) = G\boldsymbol{\Psi}\mathcal{K}(0)\boldsymbol{\Psi}^t + \eta\mathcal{I}, \qquad (1)$$

where $\boldsymbol{\Psi}$ is a propagator matrix, $\mathcal{I}$ is the identity matrix, and η equals half the average ASE noise power per frequency mode. We compute $\boldsymbol{\Psi}$ using numerical differentiation, specifically the Lyapunov method [6]. We first let $u_0(t,0)$ and $u_0(t,L)$ be the noise-free optical field at the beginning and end of the fiber span, respectively. We then perturb $u_0(t,0)$ in a single frequency mode k by a small amount Δ and launch the perturbed signal $u^{(k)}(t,0) = u_0(t,0) + \Delta\exp(i\omega_k t)$. At $z = L$, we obtain $u^{(k)}(t,L)$ by solving our nonlinear transmission equation and calculate the deviation $\delta u^{(k)} = u^{(k)}(t,L) - u_0(t,L)$ and its Fourier space vector $\boldsymbol{a}^{(k)}$. The elements of $\boldsymbol{\Psi}$ are given by $\boldsymbol{\Psi}_{jk} = a_j^{(k)}/\Delta$. We find that the Lyapunov method is numerically stable and its results are independent of the value of Δ over several orders of magnitude. By successive application of (1), we can propagate the covariance matrix from amplifier to amplifier.

The next step is the separation of the phase and timing jitter. Each pulse l in the signal has a different phase φ_l and central time τ_l. Pulses do not overlap in our test system; hence, these pulse phases evolve independently and must be removed separately from each other. We decompose the signal $u_0(t)$ into a sum of the four marks, writing $u_0(t) = \sum_{l=1}^{4} u_l(t)$, where $u_l(t) = u_0(t)$ within the bit slot of the l-th mark, and $u_l(t) = 0$ otherwise. We consider the 8 modes $iu_l(t)$ and $\partial u_l/\partial t$ and their real $2N$-dimensional Fourier space vectors $\boldsymbol{v}_l$ and $\boldsymbol{w}_l$ respectively. Note that $(\boldsymbol{v}_l, \boldsymbol{w}_l) \neq 0$ for a general signal $u_0(t)$, where $(\cdot,\cdot)$ denotes the scalar product between real vectors.[2] Consider the transformed covariance matrix

$$\overline{\mathcal{K}} \equiv \mathcal{R}^t\mathcal{K}\mathcal{R}, \qquad (2)$$

[1] In refs. [2] and [3], ω_k should be replaced by $-\omega_k$ everywhere.

[2] Except for specially constructed examples, $(\boldsymbol{v}_l, \boldsymbol{w}_l)$ vanishes only in the case of even pulses.

where $\mathcal{R}$ is an orthonormal matrix, and $\mathcal{K}$ is the covariance matrix at the end of the transmission line. We construct $\mathcal{R}$ so that the modes $\boldsymbol{v}_l$ form the first 4 columns, and the modes $\boldsymbol{w}_l$ form the following 4 columns. We fill the remaining $2N-8$ columns with columns of the $2N$-dimensional identity matrix. Next, we make $\mathcal{R}$ orthonormal by using the Gram-Schmidt procedure [7]. The phase jitter of pulse l is given by $\sigma_{\varphi,l} = \sqrt{\overline{\mathcal{K}}_{ll}}/|\boldsymbol{v}_l|$. The jitter in any other mode can be computed analogously. In the DMS system, we find a relative amplitude jitter of 17.4%, a phase jitter of $\sigma_\varphi = 2.45\pi$, and a timing jitter of $\sigma_\tau = 2.19$ ps. These values are the standard deviations of the fluctuations averaged over the four pulses and agree with our traditional Monte Carlo simulations.

We separate the phase and timing jitter by computing the matrix $\overline{\mathcal{K}}^{(r)}$ that equals $\overline{\mathcal{K}}$, except that the first 8 rows and columns are set to zero. Then we invert the transformation (2), yielding the matrix $\mathcal{K}^{(r)} \equiv \mathcal{R}\overline{\mathcal{K}}^{(r)}\mathcal{R}^t$. Using $u_0(t)$ and $\mathcal{K}^{(r)}$, we compute the pdf of the electrical narrow-band filtered receiver voltage [2].

In order to avoid roundoff errors, we compute $\overline{\mathcal{K}}^{(r)}$ by separating the jitter at every amplifier, rather than only once at the end. To each of the noise vectors $\boldsymbol{a}^{(k)}$ we apply the 2-step Gram-Schmidt orthogonalization procedure [8] to obtain residual noise vectors $\widetilde{\boldsymbol{a}}^{(k)}$ given by

$$\boldsymbol{w}' \equiv \boldsymbol{w} - \frac{(\boldsymbol{w},\boldsymbol{v})}{(\boldsymbol{v},\boldsymbol{v})}\boldsymbol{v}, \quad (3a)$$

$$\widetilde{\boldsymbol{a}}^{(k)} \equiv \boldsymbol{a}^{(k)} - \frac{(\boldsymbol{a}^{(k)},\boldsymbol{v})}{(\boldsymbol{v},\boldsymbol{v})}\boldsymbol{v} - \frac{(\boldsymbol{a}^{(k)},\boldsymbol{w}')}{(\boldsymbol{w}',\boldsymbol{w}')}\boldsymbol{w}'. \quad (3b)$$

The vectors $\widetilde{\boldsymbol{a}}^{(k)}$ are now used instead of $\boldsymbol{a}^{(k)}$ to compute $\boldsymbol{\Psi}$, and hence the phase and timing jitter that are produced during the propagation are separated from the covariance matrix. The method described in (3a) and (3b) is mathematically equivalent to removing the jitter at the end of the transmission line.

We must reintroduce the effect of the timing jitter on the pdf of the electric current at the receiver. The photodiode produces the current $I = |u_0 + \delta u|^2$, which we express as $I(t,\tau) = |u_0(t+\tau) + \delta u^{(r)}(t)|^2$, where τ is a time offset due to timing jitter, and $\delta u^{(r)}(t)$ is the optical noise field described by $\mathcal{K}^{(r)}$. We first set $\tau = 0$ and compute the pdf of $I(t,\tau=0)$, and we then convolve this pdf with the pdf of τ. We have found that τ is Gaussian distributed with variance σ_τ^2 [2]. We assume that τ is independent of $\delta u^{(r)}(t)$, thereby neglecting the cross-correlations between the $\boldsymbol{w}_l$ and the other modes in $\overline{\mathcal{K}}$. Our simulations show that the correlation between τ and $I(t,\tau=0)$ is negligible, justifying this procedure.

III. Results

Fig. 1 shows the average pdfs of the receiver voltage that result from the linearization approach as solid lines in comparison with a histogram from a traditional Monte Carlo simulation, consisting of 39,000 noise realizations represented by the dots. The voltage is normalized to the mean of the pdf of the marks. The dashed lines show a Gaussian fit to the Monte Carlo data, using the mean and variance. The large deviation between the solid and dashed curves is obvious, especially in the spaces. On the other hand, the agreement between the covariance matrix method and the Monte Carlo results is excellent in the range shown.

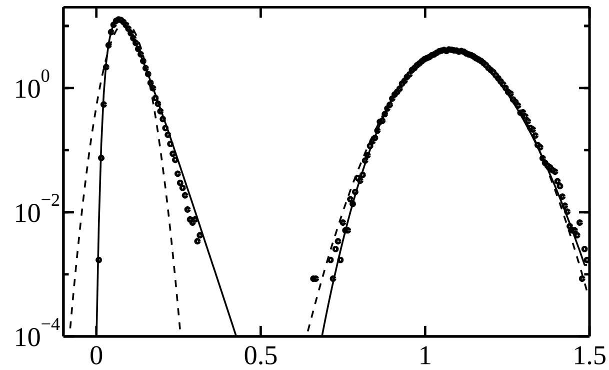

Fig. 1. Solid lines: average pdfs from the linearization approach; dots: histogram from a traditional Monte Carlo simulation; dashed lines: Gaussian fit to the dots using the mean and variance.

IV. Conclusions

In this paper, we completely describe the covariance matrix method for the first time. We apply this method to a highly nonlinear 10 Gb/s single-channel DMS system with a transmission distance of 24,000 km. Extending previous work [2], we are able to compute the pdfs of the received voltages of this system over a large range with a substantial reduction in computational time. A crucial step in this approach is the separation of the phase jitter and timing jitter, which we perform at every amplifier and separately for each mark. The computational cost of our method equals that of a Monte Carlo simulation with only $2N$ noise realizations, where N is the number of relevant complex Fourier modes; in this work we used $2N = 240$. It is our view that this approach will be of use in a wide range of commercial and experimental systems.

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