Optimization of the Split-step Fourier Method in Modeling Optical Fiber Communications Systems

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Abstract

We studied the efficiency of different implementations of the split-step Fourier method for solving the nonlinear Schrödinger equation that employ different step size selection criteria. We compared the performance of the different implementations for a variety of pulse formats and systems including higher-order solitons, collisions of soliton pulses, a single-channel periodically-stationary dispersion-managed soliton system, and chirped return to zero systems with single and multiple channels. We introduce a globally third-order accurate split-step scheme, in which a bound on the local error is used to select the step size. In many cases, this method is the most efficient when compared with commonly-used step size selection criteria, and it is robust for a wide range of systems providing a system-independent rule for choosing the step sizes. We find that a step-size selection method based on limiting the nonlinear phase rotation of each step is not efficient for many optical fiber transmission systems, although it

works well for solitons. We also tested a method that uses a logarithmic step size distribution to bound the amount of spurious four-wave mixing. This method is as efficient as other second-order schemes in the single-channel dispersion-managed soliton system, while it is not efficient in other cases including multi-channel simulations. We find that in most cases the simple approach in which the step size is held constant is least efficient of all the methods. Finally, we implemented a method in which the step size is inversely proportional to the largest group velocity difference between channels. This scheme performs best in multi-channel optical communications systems for the values of accuracy typically required in most transmission simulations.

1 Introduction

The nonlinear Schrödinger equation, which can be written as

$$i\frac{\partial u}{\partial z} - \frac{\beta''}{2}\frac{\partial^2 u}{\partial t^2} + \gamma|u|^2 u = 0, \tag{1}$$

has been shown to govern the propagation of light in a lossless optical fiber with second-order dispersion [1]. In (1), u is the complex field envelope, z is distance, β'' is the second-order dispersion, and γ is the nonlinear coefficient. The quantity $t = \tau - z/v_g$ is the retarded time, where τ is physical time and v_g is the group velocity. Even though (1) does not provide a complete physical description of a system, it is the basis for modeling optical fiber communications systems. Indeed, one can modify the nonlinear Schrödinger equation (1) to incorporate the effects of fiber loss, third-order dispersion, amplification, amplified spontaneous emission noise, and polarization mode dispersion to obtain a more realistic model of optical fiber transmission [2]–[4]. Equation (1) with $\beta'' < 0$ has a well-known analytical soliton solution with a set of remarkable properties, which gives deep insight into the nature of the dispersive and nonlinear effects [5]. However, in almost all cases (1) and its modifications cannot be solved analytically and one has to use numerical approaches. The most commonly used numerical scheme for solving (1) is the split-step Fourier method, which is convenient for its simplicity and flexibility in dealing with higher-order dispersion, the Raman

effect, and filtering [2].

In this paper we focus on the split-step Fourier method. The efficiency of the split-step method depends on both the time (or frequency) domain resolution and on the distribution of step sizes along the fiber. In simulations of optical fiber transmission systems, the time and frequency resolutions are respectively determined by the bandwidth of the signal and the number of bits that are to be propagated through the system. Consequently, the properties of the signal determine the minimum required number of Fourier modes. Although the number of Fourier modes affects the accuracy of the numerical solution, as we will discuss later, it does not change the qualitative behavior of the spatial step size selection algorithm. In this paper, we focus on the accuracy and efficiency of different spatial step size selection criteria.

A variety of step size selection criteria, most based on physical intuition, have been proposed for optimizing the split-step method. The figure of merit for each criterion is the computational cost for a given resulting global accuracy. Historically, in numerical methods used to solve (1) the step-size distribution was optimized for simulating soliton propagation. However, this optimization is not necessarily appropriate for modeling modern transmission systems, which often feature both high and low dispersion and relatively small nonlinearity, by which we mean that the nonlinear length scale is long compared to typical dispersion length scales.

The purpose of this paper is to implement an efficient, system-independent step-size selection criterion for solving the nonlinear Schrödinger equation that is based on bounding the local error, and to compare its performance to four commonly used step-size selection methods that are based on physical intuition. In the first of these four methods, called the *nonlinear phase rotation method*, the step size is chosen so that the phase change due to nonlinearity does not exceed a certain limit. This method was designed with soliton propagation in mind. The second, the *logarithmic step size method*, is designed to efficiently suppress spurious four-wave mixing, by employing a logarithmic distribution of the step sizes [6]. In the third method, the *walk-off method*, the step size is chosen to be inversely proportional to the product of the absolute value of dispersion and the spectral bandwidth of the signal. The idea behind this criterion is to resolve the collisions between pulses in different channels or at least to have a measure for the violation of this criterion. This method

was designed for low power, multi-channel systems. In the fourth, *the constant step size method*, the step sizes are kept constant along the whole transmission path.

Finally, we implement a method we call the *local error method*, in which the step size is selected by bounding the relative local error of the step. In addition, in this method we obtain a higher-order solution that is globally third-order accurate. The method is inspired by and closely related to widely-used algorithms for adaptively controlling the step size in ordinary differential equation solvers [7]. In particular, we have adopted the well-known techniques of step-doubling to estimate the local error and linear extrapolation to obtain the higher-order solution. To the best of our knowledge, although they are widely used in other fields, techniques such as these have not been previously used in simulations of optical fiber transmission systems or even seriously investigated. As is typically the case for higher-order schemes, our scheme has the advantage that it is much more computationally efficient than a second-order scheme when the global accuracy is high [7], [8]. On the other hand, it can be less efficient at low accuracy. This behavior is consistent with the results of Fornberg and Driscoll [9], who compared split-step methods of order 2, 4, and 6 with several higher-order linear multistep methods. For a two-soliton collision, Fornberg and Driscoll showed that for the global error range of 10^{-3}–10^{-2}, the second-order split-step scheme is more efficient than the fourth- and sixth-order schemes. However, for global errors smaller than 10^{-4}, the higher-order schemes become more efficient. We found similar qualitative behavior for the second-order schemes and third-order local error method that we study here.

For typical realistic optical fiber transmission systems, we will demonstrate that the walk-off method is the most efficient of the four methods in the range of accuracy of commercial interest. Nevertheless, the local error method is still competitive in this accuracy range, and, moreover, it is robust for a wide range of systems.

The remainder of the paper is organized as follows: First, we review the split-step method and the error associated with it. Then we describe the five implementations of the split-step method. Next, we discuss simulation results comparing the performance of these implementations. Finally, the conclusion follows.

2 Theory

2.1 Origin of the Split-Step Error

To estimate the local and global errors in the split-step Fourier method it is convenient to represent (1) in the form

$$\frac{\partial u(z,t)}{\partial z} = (\hat{D}+\hat{N}[u])u(z,t), \tag{2}$$

where $\hat{D} = -i(\beta''/2)\partial^2/\partial t^2$ is the dispersion operator and $\hat{N}[u] = i\gamma|u|^2$ is the nonlinear operator. Although the following discussion is for the nonlinear Schrödinger equation (1), the arguments and conclusions also apply to the modified versions of (1) that model realistic optical fiber transmission systems and to general reaction-diffusion equations. In the symmetric split-step scheme, the solution to (2) is approximated by

$$u(z+h,t) \approx \exp\left(\frac{h}{2}\hat{D})\right)\exp\left\{h\hat{N}\left[u\left(z+\frac{h}{2},t\right)\right]\right\}\exp\left(\frac{h}{2}\hat{D}\right). \tag{3}$$

Since the dispersion and nonlinear operators do not commute in general, the solution (3) is only an approximation to the exact solution. An argument based on the Baker-Campbell-Hausdorff formula shows that the *local error*, which is the error incurred in a single step of the symmetric split-step scheme, has a leading order term which is of third order in the step size h, *i.e.*, the error is $O(h^3)$ [10]. When we state that an error is $O(h^m)$ we mean that it is bounded by Ch^m, for some constant C. Since the total number of steps in a fiber span is inversely proportional to the average step size, the *global error* accumulated over a fiber span is second order in the step size, $O(h^2)$.

Finding an optimal step size distribution depends on the particular optical transmission system. We will review several criteria for choosing the step size in the split-step Fourier method, and we will introduce a new criterion based on a measure of the local error.

2.2 Nonlinear Phase Rotation Method

The nonlinear phase rotation method is a variable step size method that is designed for systems in which nonlinearity plays a major role. For a step of size h, the effect of the nonlinear operator $\hat{N}$ is to increment the phase of u by an amount $\phi_{\mathrm{NL}} = \gamma|u|^2 h$. If we impose an upper limit $\phi_{\mathrm{NL}}^{\max}$ on the nonlinear phase increment ϕ_{NL}, we obtain the bound on the step size:

$$h \leq \frac{\phi_{\mathrm{NL}}^{\max}}{\gamma|u|^2}. \tag{4}$$

This criterion for selecting the step size was originally applied to simulate soliton propagation and is widely used in optical fiber transmission simulators. However, as we will show later, this approach is far from optimal for many modern communications systems.

2.3 Spurious Four-Wave Mixing and Logarithmic Step Size Distribution

An improper distribution of the step sizes may lead not only to a general reduction of accuracy, but also to numerical artifacts. Forghieri [11] demonstrates that the power of the four-wave mixing products can be greatly overestimated by a constant step size method, since four-wave mixing is a resonance effect. To efficiently suppress this numerical artifact, Bosco, *et al.* [6] used a logarithmic distribution of the step sizes to keep the spurious four-wave mixing components below a certain level. For a fiber span of length L and loss coefficient Γ, the step size of n-th step is given by

$$h_n = -\frac{1}{2\Gamma} \ln\left[\frac{1-n\sigma}{1-(n-1)\sigma}\right], \tag{5}$$

where $\sigma = \left[1-\exp(-2\Gamma L)\right]/K$, and K is the number of steps per fiber span. We will call this implementation of the split-step method the *logarithmic step size method.*

2.4 Walk-off Method

In many optical fiber communications systems chromatic dispersion is the dominant effect, and nonlinearity only plays a secondary role, particularly in multi-channel systems in which the wave-

length channels cover a broad spectrum. In this case it can be reasonable to use the *walk-off method*, in which the step size is determined by the largest group velocity difference between channels. The basic idea is to choose the step size to be smaller than a characteristic walk-off length. In a multi-channel system with large local dispersion, pulses in different channels move through each other very rapidly. To resolve the collisions between pulses in different channels, the step size in the walk-off method is chosen so that in a single step two pulses in the two edge channels shift with respect to each other by a time which is a specified fraction of the pulse width. Consequently, the step size is given by

$$h = \frac{C}{\Delta V_{\mathrm{g}}}, \tag{6}$$

where ΔV_{g} is the largest group velocity difference between channels and C is a constant that can vary from system to system. In any system, $\Delta V_{\mathrm{g}} = \left| D_2\lambda_2 - D_1\lambda_1 \right|$, where D_1 and D_2 are the dispersions corresponding to the smallest and largest wavelengths λ_1 and λ_2. Since ΔV_{g} is constant in any particular kind of fiber, in a given type of fiber the step size is constant. The walk-off method can be applied to single-channel as well as multi-channel systems by choosing λ_1 and λ_2 at the two edges of the signal spectrum.

2.5 Constant Step Size Method

The simplest way to implement the split-step Fourier method is to use a constant step size along the whole transmission path. The global accuracy can be improved only by increasing the total number of steps. Note that the walk-off and constant step size methods are identical in systems with only one type of fiber.

2.6 Local Error Method

In practice, it is desirable to have a general criterion for choosing the step size distribution that is close to optimal for an arbitrary system. Adaptive methods for controlling the step size using a measure of the local error are widely used in ordinary differential equation solvers [7]. We

have implemented a scheme based on bounding the error in each step using the technique of step-doubling and local extrapolation. Given the field u at a distance z, our aim is to compute the field at $z+2h$. Suppose that we perform one step of size $2h$ in a symmetric split-step scheme. We will refer to the solution obtained at $z+2h$ as the *coarse solution*, u_c. Since the local error in the symmetric split-step scheme is third order, there is a constant κ so that

$$u_c = u_t + \kappa(2h)^3 + O(h^4), \tag{7}$$

where the *true solution* u_t is the exact solution at $z+2h$ obtained from the given solution at z. When we write that $u = v + O(h^4)$ for some functions u and v, we mean that $|u-v| < Ch^4$, for some constant C. Next, we return to z and compute the *fine solution* u_f at the same distance $z+2h$ using two steps of size h. As above, the fine solution is related to the true solution by

$$u_f = u_t + 2\kappa h^3 + O(h^4). \tag{8}$$

By taking an appropriate linear combination of the fine and coarse solutions we can obtain an approximate solution at $z+2h$ for which the leading order error term is of fourth order in the step size h [7]. From (7) and (8) it follows that this higher-order solution is given by

$$u_4 = \frac{4}{3}u_f - \frac{1}{3}u_c = u_t + O(h^4), \tag{9}$$

which we take as the input to the next step of size $2h$.

In the local error method the step size is adaptively chosen so that the local error incurred from z to $z+2h$ is bounded within a specified range. Now the relative local error δ_4 of the higher-order solution is defined by

$$\delta_4 = \frac{\|u_4 - u_t\|}{\|u_t\|}, \tag{10}$$

where the norm $\|u\|$ is defined as $\|u\| = \left(\int |u(t)|^2 dt\right)^{1/2}$. However, since we cannot compute the

true solution u_t in practice, we cannot compute the local error using (10). Instead, we define the *relative local error* of a step to be the local error in the coarse solution relative to the fine solution:

$$\delta = \frac{\|u_f - u_c\|}{\|u_f\|}. \tag{11}$$

Notice that δ is a measure of the true local error δ_4, since δ can be obtained from $3\delta_4$ by replacing u_t by u_f. The step size is chosen by keeping the relative local error δ within a specified range $(1/2\delta_G, \delta_G)$, where δ_G is the goal local error. If $\delta > 2\delta_G$, the solution is discarded and the step size is halved. If δ is in the range $(\delta_G, 2\delta_G)$, the step size is divided by a factor of $2^{1/3}$ for the next step. If $\delta < 1/2\delta_G$, the step size is multiplied by a factor of $2^{1/3}$ for the next step. The reason for choosing this factor is that then the local error should increase by a factor less than 2, since it is proportional to h^3.

Rather than simply computing the fine solution, our method computes both the fine and coarse solutions. Although it requires 50% more Fourier transforms than does the standard symmetric split-step method, the method yields both a higher-order solution, which is globally third-order accurate, and a measure of the relative local error which is used to control the step size. However, it is important to understand that the higher-order solution u_4 is not always more accurate than the fine solution u_f, especially when the step size is large, since we are bounding the local error δ of the coarse solution relative to the fine solution, rather than the true local error δ_4 of the scheme.

Since we do not make any assumptions about the physical properties of the system, such as the amount of nonlinearity or dispersion, we expect the local error method to work well in an arbitrary system. In order to simulate a system with optimal efficiency, one first needs to investigate it to ascertain the major sources of the split-step error. Assuming that the system is dominated by one source of error, one can select an appropriate criterion for choosing the step sizes. The local error method allows us to deal with general systems when the major source of error is unknown or may even change during the propagation, or when performing a series of simulations in which the system parameters are varied. The method can be applied to a variety of systems without sacrificing too much computational efficiency.

3 Numerical Results

In this section we compare the efficiency of the five implementations of the split-step method described in Section 2. Since most of the computational time is consumed by evaluating fast Fourier transforms (FFTs), we use the number of FFTs per simulation as a measure of the total computational cost [9]. We used the following scheme to compare the different methods. First, we compute a solution u_a that is accurate to machine precision using the standard symmetric split-step method (with step sizes on the order of 5 cm). Next we compute the numerical solution u_n for each of the different split-step implementations, and calculate the *global relative error* ε defined by

$$\varepsilon = \frac{\|u_n - u_a\|}{\|u_a\|}, \tag{12}$$

where we use the norm defined in Section 2.6. We compare the performance of the different methods by plotting the number of FFTs versus the global relative error.

3.1 Higher-Order Solitons

We start with the propagation of second- and fifth-order solitons. These systems are both highly nonlinear. In addition, higher-order solitons are very sensitive to numerical errors, thus requiring an efficient adaptive algorithm. The exact functional form of the N-soliton solution can be found in [3], [5]. We use an anomalous-dispersion fiber with $\beta'' = -0.1$ ps^2/km. The initial pulse is a hyperbolic secant of the form $u(t) = A\eta\left(|\beta''|/\gamma\right)^{1/2}\mathrm{sech}(\eta t)$, where the nonlinear coefficient is $\gamma = 2.2$ W^{-1}km^{-1}, the inverse pulse duration is $\eta = 0.44$ ps^{-1}, and where $A = 2$ and $A = 5$ for the second-order and fifth-order solitons respectively. The corresponding FWHM pulse duration is 4 ps and the peak powers are 35 mW and 220 mW for the second- and fifth-order solitons respectively. The number of Fourier modes is 1024 and the simulation time window is 50 ps. We show the performance of the different implementations of the split-step method applied to the second-order soliton in Fig. 1(a) and to the fifth-order soliton in Fig. 1(b). In Fig. 1, we have plotted the number of FFTs versus the global relative error for the different step-size criteria. Although the performance of the local error method is not significantly better in the range of low accuracy values

10^{-2}–10^{-3}, at high accuracy the computational cost of the local error method is one or two orders of magnitude less than for other methods. Notice that the nonlinear phase method performs better than the constant step size method, consistent with the system's large nonlinearity. The slope of the local error method curve is less than those of the other two methods since the constant step size and nonlinear phase methods are globally second-order accurate, while the local error method is globally third-order accurate. The walk-off and constant step size methods are identical since this system includes only one type of fiber. The logarithmic step size method reduces to the constant step size method because the fiber is lossless, and (5) leads to a constant step size distribution.

3.2 Soliton Collisions

Soliton collisions can be a good test for numerical methods because the subtle effect of four-wave mixing cancellation after the collision is very sensitive to numerical errors [3]. The fiber type and the initial pulse shape are the same as in Sec. 3.1, except that $A = 1$. The pulse duration is 4 ps and the peak power is 8.8 mW. We launch two soliton pulses separated in time by 100 ps and with a central frequency difference of 800 GHz. The number of Fourier modes is 3072 and the simulation time window is 400 ps. We show the performance of the different methods in Fig. 2. The local error, constant step size, and nonlinear phase rotation methods perform equally well at low accuracy, when the global error is in the range 10^{-3}–10^{-1}, while the local error method is much more efficient when the global error is less than 10^{-4}. Global errors less than 10^{-4} are required to estimate the four-wave mixing terms correctly and to have them cancel out after the collision. The nonlinear phase method still works better than the constant step size method because the nonlinear interactions are critical in the propagation. As in Section 3.1, the logarithmic step size and walk-off methods reduce to the constant step size method.

Using the example of a soliton collision, we illustrate the adaptive behavior of the local error algorithm. Fig. 3 shows the step size as a function of propagation distance for the soliton collision when the targeted range for the local error is (0.5×10^{-5}, 10^{-5}) and the initial guess for the step size is 1000 m. Since the local error for this initial step is much less than the targeted range of

values, at each step the step sizes are increased until the local error is within the targeted range. The pulse collision occurs at a distance of 200 km. At this point, we observe a significant decrease in the step size, which is necessary to accurately resolve the collision. After the collision, the step size is increased to the same value as before the collision. The last step is smaller than the previous step simply because the remaining section of the fiber is shorter than the step size chosen by the algorithm.

3.3 Single-Channel Systems

In this section we study periodically-stationary dispersion-managed soliton (DMS) and chirped-return-to-zero (CRZ) systems that resemble experimental systems [12], [13]. The DMS system is highly nonlinear, meaning that both dispersion and nonlinearity determine the signal evolution, while the CRZ system is quasilinear and the evolution is mostly determined by dispersion [14], [15]. Thus we are studying the four split-step implementations using two different types of systems. We include fiber attenuation and gain, but we do not consider amplifier noise. We use 64-bit random bit streams that repeat periodically. We stress that our goal is to test the performance of the numerical methods for realistic systems rather than to achieve optimal propagation. Consequently, it is important that we have pulse streams rather than single pulses, that we use dispersion management, and that we include the effects of fiber loss and amplifier gain.

The DMS system is based on a 107 km dispersion map, which consists of four dispersion-shifted fiber spans, each of 25 km, with normal dispersion equal to -1.10 ps/nm-km, followed by 7 km of standard single-mode fiber with anomalous dispersion of 16.6 ps/nm-km at 1551 nm [12]. The loss in both fibers is 0.21 dB/km, and the amplifier spacing is 25 km with an additional amplifier after the standard single-mode fiber. We use Gaussian pulses with a FWHM duration of 9 ps, as is appropriate for a 10 Gbit/s bit rate. The peak power is 8 mW. The signal is launched in the middle of a span of anomalous fiber to ensure the periodicity of the pulse shape as it propagates along the fiber. The propagation distance is 1,280 km. The simulation time window is 6400 ps and the number of Fourier modes is 6144. We have not included a dispersion slope in this system since there is only a single channel and previous work indicates that higher-order dispersion plays

no role [12].

The CRZ system is based on a 180 km dispersion map consisting of 160 km of dispersion-shifted fiber with dispersion -2.44 ps/nm-km, followed by 20 km of standard fiber with dispersion 16.55 ps/nm-km [13]. The dispersion slope is 0.075 ps^2/nm-km and the fiber loss is 0.21 dB/km for both fibers, while the amplifier spacing is 45 km. Symmetric dispersion pre- and post-compensation is performed using fiber spans of length 2.0 km, where the dispersion is 93.5 ps/nm-km, the slope is -0.2 ps^2/nm-km and the loss is 0.5 dB/km. The initial pulses are phase-modulated, raised-cosine pulses with 1 mW peak power and a chirp parameter equal to -0.6 [14]. The bit rate is 10 Gbit/s and the propagation distance is 1,800 km. The simulation time window is 6400 ps and the number of Fourier modes is 4096.

The performance of the four split-step implementations for the single-channel DMS and CRZ systems is shown in Figs. 4(a) and (b) respectively. In both systems, the local error method performs best over the entire range. Due to its higher order of accuracy, the data points for the local error method lie on a line with a smaller absolute slope than those of the other methods, as expected. However, all methods become comparable in the range of global errors 10^{-3}–10^{-1}, the region of most interest in simulating fiber optic links. We note however, that in the CRZ system the performance of the logarithmic step size method is somewhat poorer than that of the nonlinear phase and walk-off methods.

3.4 Multi-Channel CRZ System

In order to compare the split-step implementations for modeling multi-channel communications systems, we used the same CRZ system as described in Section 3.3. In Fig. 5, we show the performance of the split-step selection criteria on a 5-channel CRZ system with a 50 GHz channel spacing. As in the single-channel case, the local error method is much more efficient at high accuracy. However, at low accuracy, with the global error in the range 10^{-3}–10^{-1}, which is typical for most practical systems, the walk-off method performs best. At low accuracy, the local error method does not perform as well as the walk-off method for the following reasons. First, in the multi-channel CRZ system, the step size within each fiber in the local error method varies approx-

imately within a factor of two, and the average value is comparable to the step size in the walk-off method for a given global error. However, each pair of steps in the local error method is 50% more expensive than in the walk-off method. In addition, when the step size is large and the global accuracy is low, the higher-order solution u_4 may not be as accurate as the fine solution u_f. Indeed, we have observed that the local error method performs slightly better at low global accuracy if we keep the fine solution u_f instead of the higher-order solution u_4 at each step.

Next, we observe that the nonlinear phase rotation method does not perform as well as the walk-off method in the multi-channel CRZ system, although the performance of the two methods is comparable in the single-channel DMS and CRZ systems. There are two major reasons for this behavior. First, in contrast to the single-channel case, the walk-off criterion becomes more physically relevant in a WDM system, in which pulses in different channels collide. Second, the step size in the nonlinear phase rotation method is determined by the peak power in the time domain. In the single-channel CRZ system, the power function contains spikes due to the overlap between neighboring pulses. However, between amplifiers the peak power decreases monotonically with distance due to fiber attenuation. By contrast, the peak power of the multi-channel system does not decrease monotonically with distance but contains irregular spikes because pulses from different channels rapidly pass through each other. As a consequence, there is a significant proportion of step sizes in the nonlinear phase rotation method that are much smaller than they need to be for a given global accuracy. The logarithmic step size method is not efficient in the CRZ system because the step size choice is only based on limiting spurious four-wave mixing, which is only one of the potential sources of error in a multi-channel simulation. We also found that in the logarithmic step size method, the error grows most rapidly in fibers with high dispersion. We find that the constant step size method is inefficient in the multi-channel CRZ system. The reason it performs so poorly is that for a given step size the global error does not accumulate linearly with distance. Consequently, in some sections of the transmission line the global error grows rapidly, while in others the error accumulates very slowly and computational effort is wasted.

In Figure 6, we show the step sizes in the local error method as a function of propagation distance when the targeted range for the local error is $(0.5 \times 10^{-4}, 2 \times 10^{-4})$. The upper two

plots show the step sizes for the first two and last two periods of the dispersion map, and the lower two plots show the corresponding portions of the dispersion map. The amplifiers, marked by triangles, are placed after the pre-compensation fiber, and then every 45 km. Notice that the step size increases as the signal power and the strength of the nonlinear interactions decrease due to the fiber loss. Also note that step size is smaller in fibers with higher dispersion since the pulses in neighboring channels move faster with respect to each other.

To characterize the dependence of the local error method on the number of Fourier modes, we modeled the multi-channel CRZ system with different numbers of Fourier modes, N. The results in Fig. 5 were obtained with $N = 4096$. We also simulated the system with $N = 3072$, $N = 6144$, and $N = 8192$. In each case, the log-log plots of the number of FFTs versus the global error have the same slope as for the local error method plot shown in Fig. 5. However, the curves are slightly shifted with respect to each other. The number of FFTs required to achieve a given global error can increase by a factor of two as N is increased from 3072 to 6144. Since we keep the time window fixed, the frequency window increases as N increases, and more high-frequency components contribute to the global error. However, if we further increase N from 4096 to 8192, the number of FFTs grows by less than 30%, since the additional high frequency modes lie well outside the bandwidth of the WDM signal and contribute little to the error.

3.5 Variation of Method Parameters

In this section we address two important questions concerning how the method parameter should be chosen to achieve a desired global accuracy. The method parameter is the parameter in a split-step method that we vary to adjust the accuracy of the method. First, for a given global error, how much does the method parameter depend on the particular system? Second, by what factor should the method parameter be decreased to halve the global error?

To answer the first question, in Figs. 7(a)–(e), we show the dependence of the global error on the method parameter for the local error, walk-off, nonlinear phase, logarithmic step, and constant step size methods respectively. Although the walk-off method is the most efficient in some cases, it exhibits the worst system dependence. In particular, for the five systems we studied, when

the global error is 10^{-3} the walk-off parameter varies over five orders of magnitude, whereas the parameter for the other three methods vary only over one to two orders of magnitude. Even omitting the two soliton systems from the comparison, the walk-off method has a greater system dependence than the local error method. Consequently, each new system requires a significantly different walk-off parameter to achieve the same global accuracy.

To answer the second question we examine the slopes of the curves in Fig. 7. For the walk-off, nonlinear phase, and logarithmic steps methods, the slopes are approximately 2, as expected, since these three schemes are second order and the step sizes depend linearly on the method parameter. Ideally, the global error should depend linearly on the local error. However, for the local error method the slopes of the curves in Fig. 7(a) are approximately 1.3, rather than 1. The reason for this discrepancy is that the true local error (10) is unavailable. Instead, we use an estimate of local error given by (11). In addition, in our local error algorithm, the local error (11) is maintained within a range of values rather than being kept constant.

4 Conclusion

We have studied the performance of different implementations of the symmetric split-step Fourier method for solving the nonlinear Schrödinger equation applied to various optical fiber transmission systems. We developed an implementation of the symmetric split-step Fourier method that is globally third-order accurate, and for which the step sizes are chosen using a criterion that keeps the local error within a specified range. We showed that the local error method performs best for modeling optical solitons, soliton pulse interactions, and single-channel transmission systems. Because it is a higher-order method, the local error method is much more computationally efficient at high accuracy than the other three methods we considered for all of the systems we studied. This behavior is expected with higher-order schemes [9]. Moreover, even at low accuracy, the local error method has the advantage that it is robust for arbitrary systems.

We find that the nonlinear phase rotation method is inefficient for modeling typical modern optical fiber transmission systems, although it performs reasonably well for solitons. The logarithmic step size method, which is based on bounding the spurious four-wave mixing in each step [6], and the constant step size method are not efficient for multi-channel systems, although they can be as efficient as the nonlinear phase and walk-off methods in single-channel systems. Finally, the walk-off method, in which the step size is chosen to be inversely proportional to the fiber dispersion, performs well for multi-channel systems over the accuracy range of interest in commercial applications.

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Figure Captions

1. Plot of the total number of FFTs versus global relative error ε for (a) second-order and (b) fifth-order solitons.

2. Plot of the total number of FFTs versus global relative error ε for a collision of two first-order solitons.

3. Step size *h* as a function of distance for the local error method applied to a collision of two first-order solitons.

4. Plot of the total number of FFTs versus global relative error ε for the single-channel (a) DMS and (b) CRZ systems.

5. Plot of the total number of FFTs versus global relative error ε for the multi-channel CRZ system.

6. Step size *h* as a function of distance for the local error method applied to the multi-channel CRZ system. The upper two plots show the step sizes for the first two and last two periods of the dispersion map, and the lower two plots show the corresponding portions of the dispersion map. Triangles indicate the positions of amplifiers.

7. Plot of the global error as a function of method parameter for (a) local error, (b) walk-off, (c) nonlinear phase, (d) logarithmic step, and (e) constant step methods.

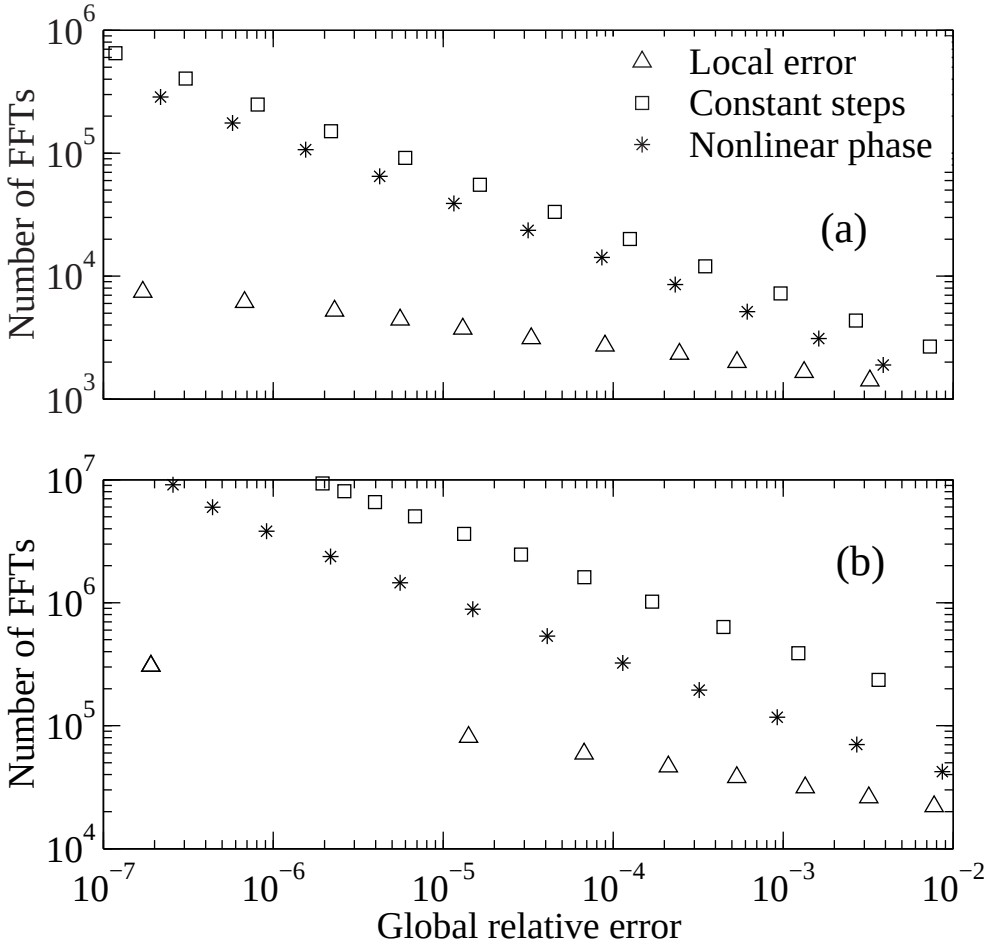


Figure 1: Sinkin, *et al.*

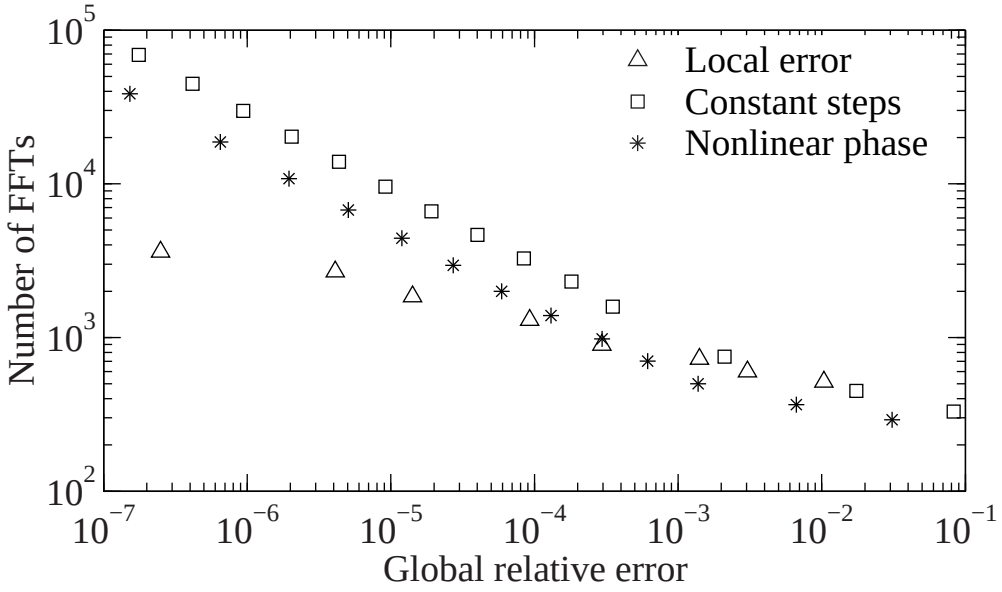


Figure 2: Sinkin, *et al.*

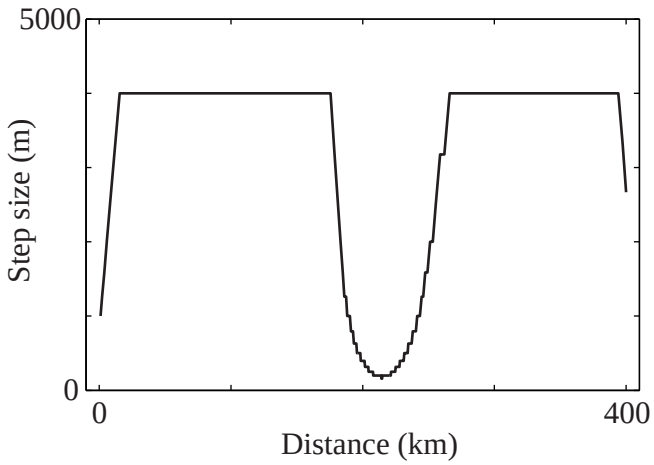


Figure 3: Sinkin, *et al.*

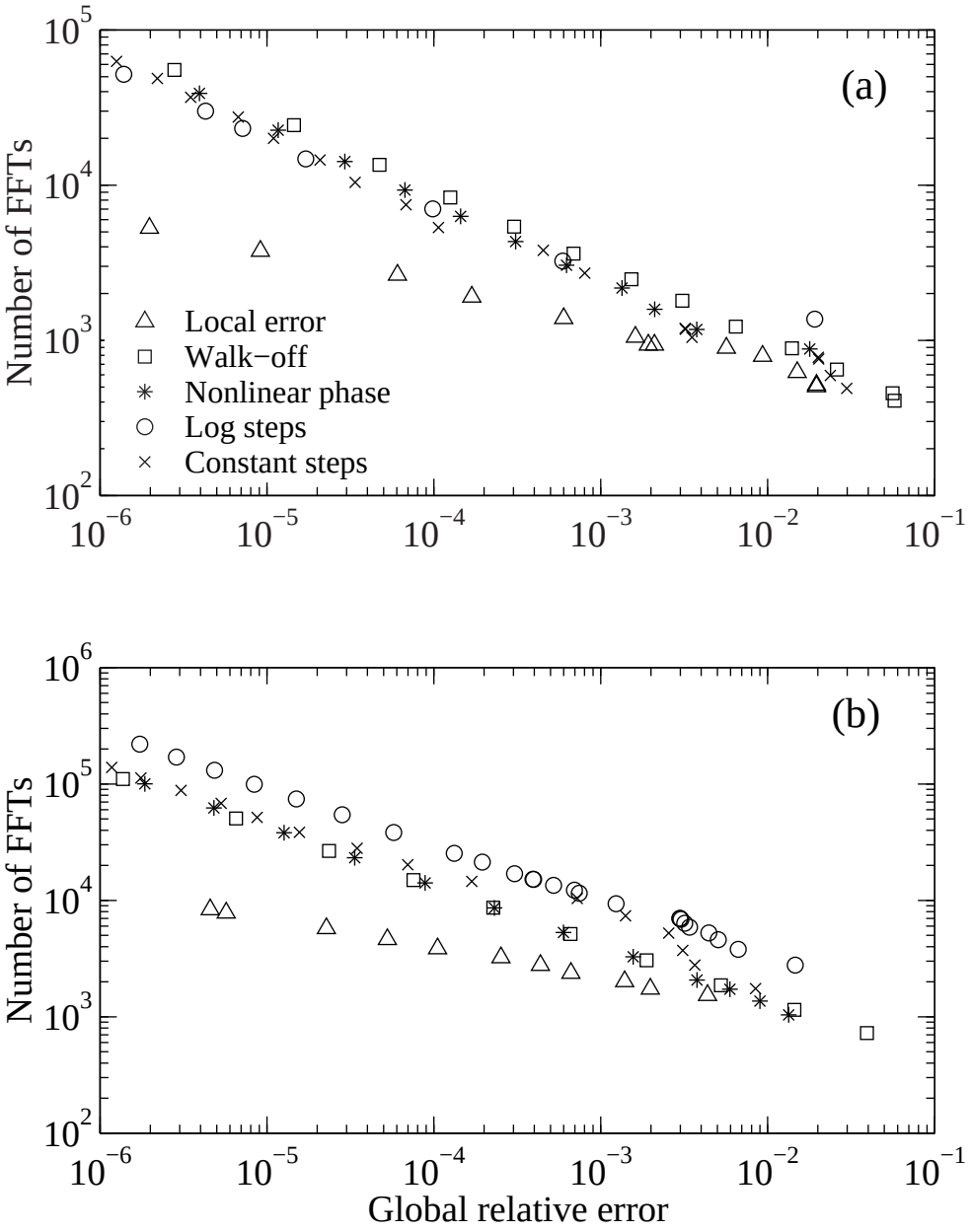
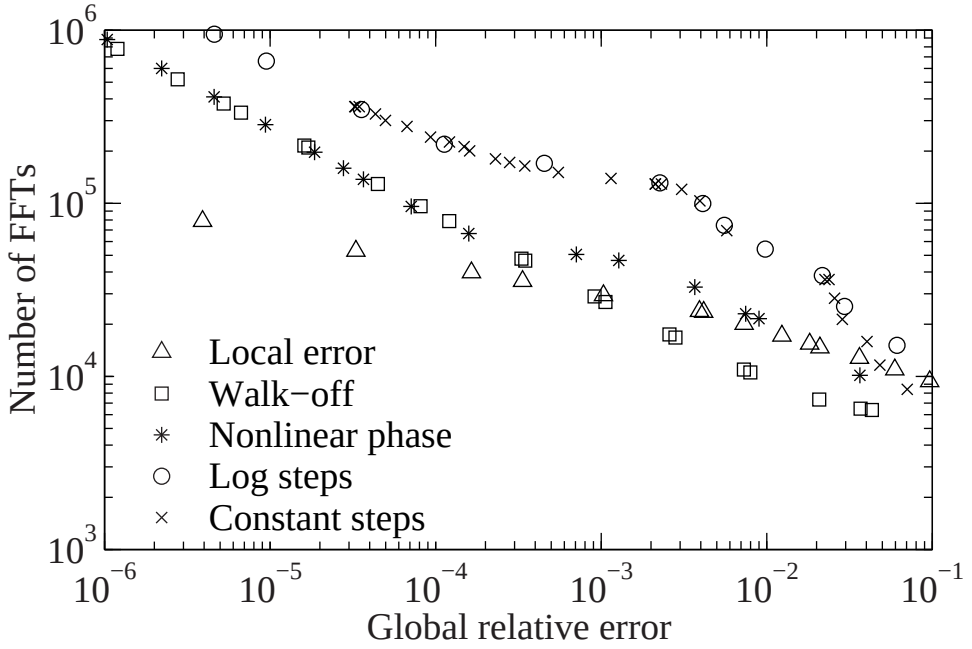


Figure 4: Sinkin, *et al.*



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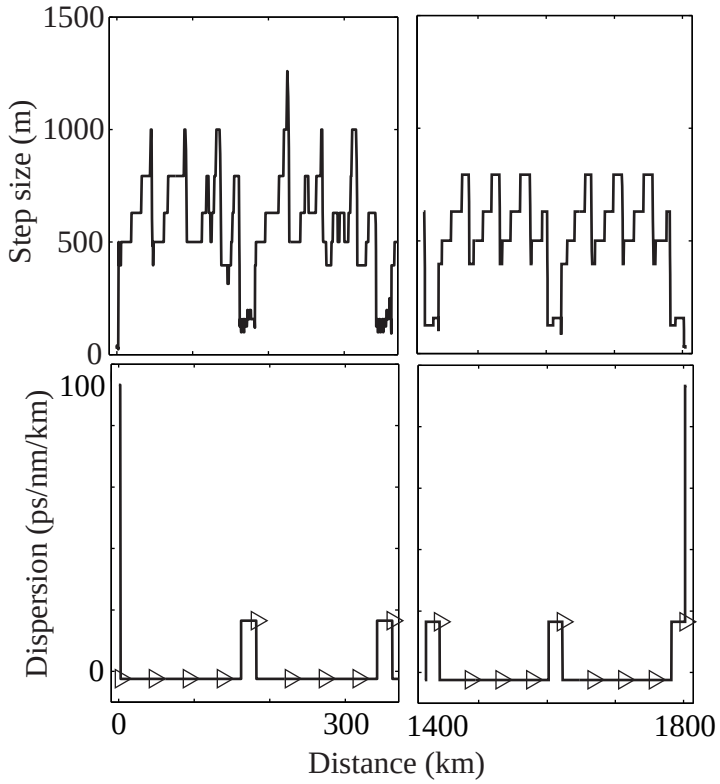

Figure 6: Sinkin, *et al.*

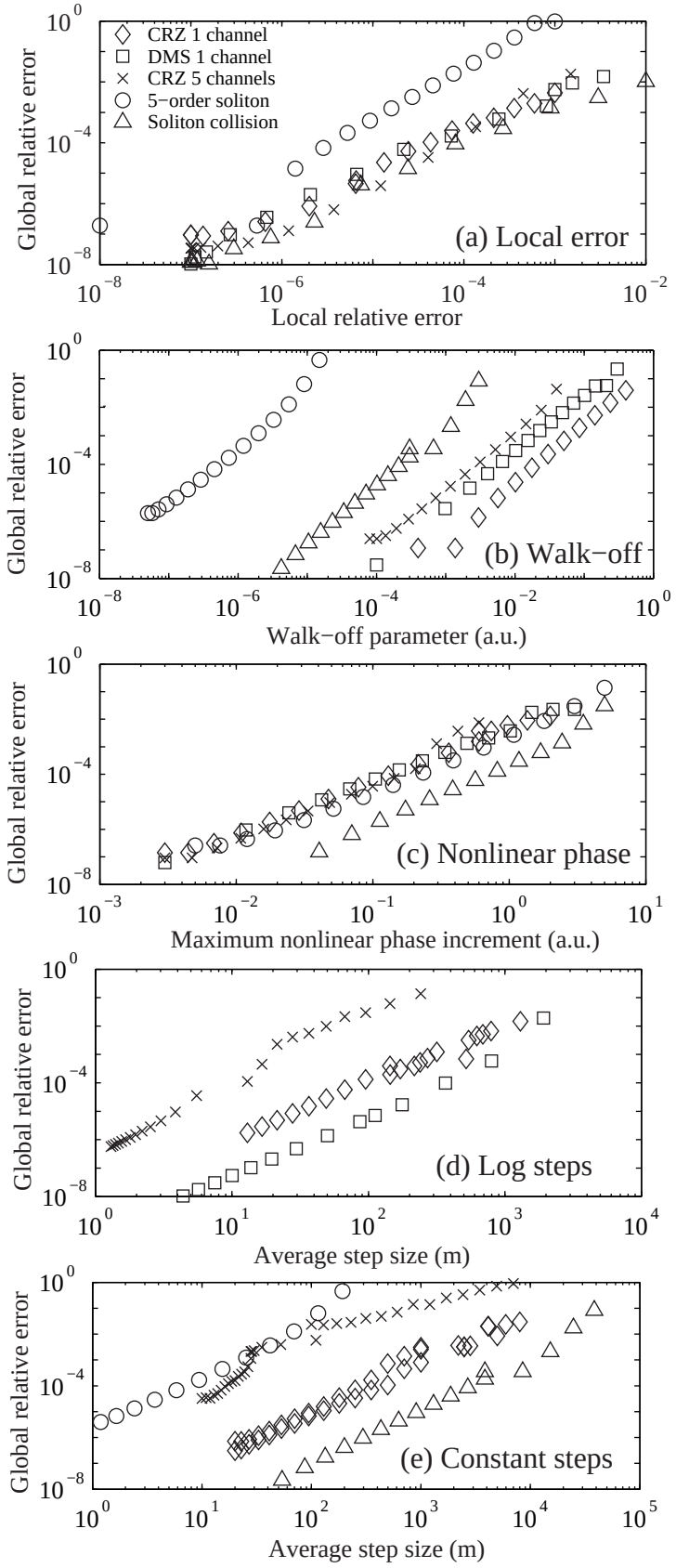

Figure 7: Sinkin, *et al.*