MATH 423 (Spring 2012) Exam II, April 12th

No calculators, books or notes!
Show all work and give complete explanations for all your answers.

(1) [12 pts]
(a) Define the concept of a 1-form on \( \mathbb{R}^3 \).

(b) Let \( V \) be a vector field on \( \mathbb{R}^3 \). Define \( \omega \) by \( \omega_p(w) = w \cdot V(p) \), where \( p \) is a point in \( \mathbb{R}^3 \) and \( w \) is a tangent vector to \( \mathbb{R}^3 \) at \( p \). Prove that \( \omega \) is a 1-form on \( \mathbb{R}^3 \).
(2) [16 pts]
(a) Define the concept of a coordinate patch.

(b) Does the equation \((x^2 + y^2)^2 + 4z^2 - 16z + 15 = 0\) define a surface?
(3) [12 pts] Let $M$ be the surface parametrized by $\mathbf{x}(u, v) = (u \cos v, u \sin v, v)$ for $0 < u < 1$ and $0 < v < \pi$. Orient $M$ using the upward normal. Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ defined by $\mathbf{F}(x, y, z) = (y, -x, z^2)$. Calculate $\iint_M \mathbf{F} \cdot d\mathbf{A}$. 


(4) [15 pts] The catenoid, \( M \), is the surface obtained by rotating the catenary curve \( z = \cosh y \) about the \( y \)-axis. [Recall that \( \cosh x = (e^x + e^{-x})/2 \) and \( \sinh x = (e^x - e^{-x})/2 \).]

(a) Find a formula for a coordinate patch \( \mathbf{x} : D \to \mathbb{R}^3 \) for \( M \). Make sure you specify the domain \( D \) of \( \mathbf{x} \). [Do not prove that \( \mathbf{x} \) is a patch, just write down a formula for it.]

(b) Sketch \( M \) together with some of its coordinate (grid) curves.
(c) Calculate a parametrization of the tangent plane to $M$ at the point $(x, y, z) = (\frac{\sqrt{3}}{2} \cosh 1, 1, \frac{1}{2} \cosh 1)$.

(5) [8 pts] Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$. About what axis is the circulation of $\mathbf{F}$ the greatest at the point $(2, 1, 3)$?
(a) Give a careful statement of the Divergence Theorem.

(b) Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$. Let $p \in \mathbb{R}^3$ and let $B_\epsilon$ be the ball of radius $\epsilon > 0$ centered at $p$. (So $B_\epsilon$ is a three-dimensional solid.) Let $\partial B_\epsilon$ denote the boundary surface of $B_\epsilon$, oriented using the outward pointing normal vector field. Prove that

$$(\nabla \cdot \mathbf{F})(p) = \lim_{\epsilon \to 0} \frac{1}{\text{Vol}(B_\epsilon)} \int_{\partial B_\epsilon} \mathbf{F} \cdot d\mathbf{A}.$$ 

(c) Now suppose that $\mathbf{F} = \rho \mathbf{v}$ where $\rho$ is the density of a fluid and $\mathbf{v}$ is its velocity vector field. Use the result in (b) to provide a physical interpretation of $\nabla \cdot \mathbf{F}$ at $p$. 

Pledge: I have neither given nor received aid on this exam

Signature: ________________________________