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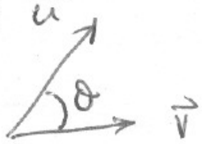
NAME : SOLUTIONS

1	20
2	15
3	15
4	15
5	10
T	75

MATH 251 (Fall 2004) Exam 1, Oct 1st

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 65 minute exam. It is worth a total of 75 points.

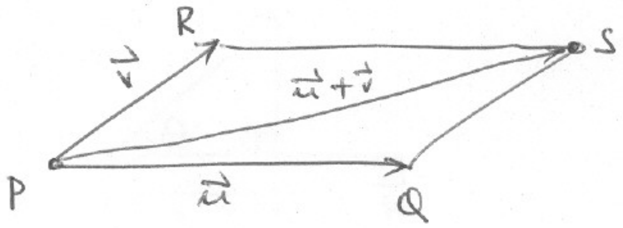
(1) [20 pts]
(a) Find the dot product of two vectors if their lengths are 6 and $\frac{1}{3}$ and the angle between them is $\frac{\pi}{4}$.



$|\vec{u}| = 6 \quad |\vec{v}| = \frac{1}{3} \qquad \theta = \pi/4$

$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta = 6 \cdot \frac{1}{3} \cos \pi/4 = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$

(b) Find the area of the parallelogram with vertices $\overset{P}{(1,2,3)}$, $\overset{Q}{(0,2,5)}$, $\overset{R}{(4,6,8)}$, and $\overset{S}{(3,6,10)}$.



$\vec{u} = \vec{PQ} = (0\ 2\ 5) - (1\ 2\ 3)$
$= (-1\ 0\ 2)$

$\vec{v} = \vec{PR} = (4\ 6\ 8) - (1\ 2\ 3)$
$= (3\ 4\ 5)$

$\vec{u} + \vec{v} = (2, 4, 7)$

$\vec{PS} = (3\ 6\ 10) - (1\ 2\ 3) = (2\ 4\ 7)$

Good $\vec{u} + \vec{v} = \vec{PS}$ ✓

So

$\text{Area} = |\vec{u} \times \vec{v}|$

$= \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 2 \\ 3 & 4 & 5 \end{vmatrix} \right|$

$= |(-8, 11, -4)| = \sqrt{64 + 121 + 16} = \sqrt{201}$

$$\begin{array}{r} 64 \\ 16 \\ 121 \\ \hline 201 \end{array}$$

(c) Find the scalar projection of the vector $\mathbf{v} = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ onto the vector $\mathbf{w} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

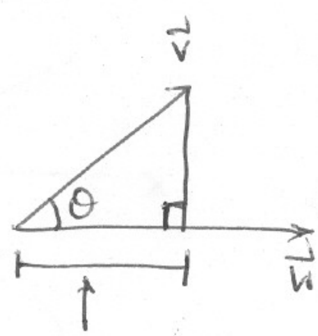
$$P_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} = \frac{(1,6,-2)\cdot(2,-3,1)}{\sqrt{2^2+3^2+1^2}}$$

$$= \frac{2-18-2}{\sqrt{14}} = \frac{-18}{\sqrt{14}}$$

$$= -\frac{18}{14}\sqrt{14} = -\frac{9}{7}\sqrt{14}$$

(d) Draw a picture and write a sentence or two that clearly explain the geometrical meaning of the scalar projection of a vector **v** onto another vector **w**.

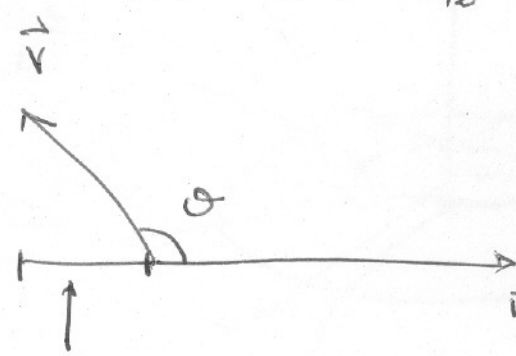
$0 < \theta < \pi/2$



LENGTH $= |P_{\vec{w}}\vec{v}|$

$P_{\vec{w}}\vec{v}$ IS POSITIVE

$\pi > \theta > \pi/2$



LENGTH $= |P_{\vec{w}}\vec{v}|$

$P_{\vec{w}}\vec{v}$ IS NEGATIVE

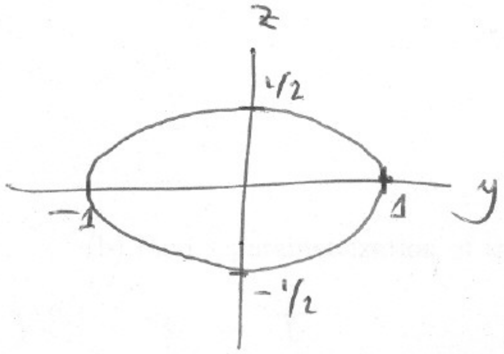
$P_{\vec{w}}\vec{v}$ is the component of $\vec{v}$ in the direction of $\vec{w}$. This component is positive if $\vec{v}, \vec{w}$ have an angle θ that is less than $\pi/2$ and negative if $\theta > \pi/2$.

(2) [15 pts] Sketch the following surfaces

(a) $y^2 - x^2 + 4z^2 = 1$. Also sketch some appropriately chosen traces (*i.e.*, slices) of this surface.

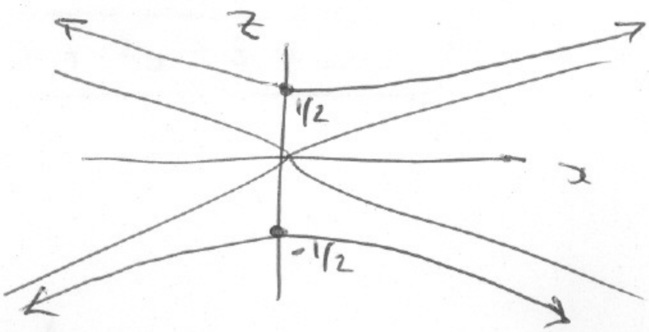
$\boxed{x=0}$ $y^2 + 4z^2 = 1$

$\left(\frac{y}{2}\right)^2 + z^2 = \left(\frac{1}{2}\right)^2$

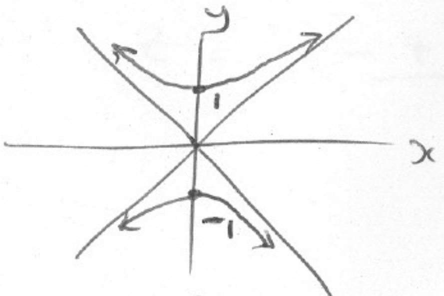


$\boxed{y=0}$ $4z^2 - x^2 = 1$

Asymptotes: $z = \pm\frac{x}{2}$

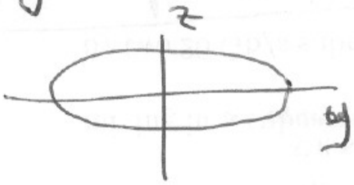


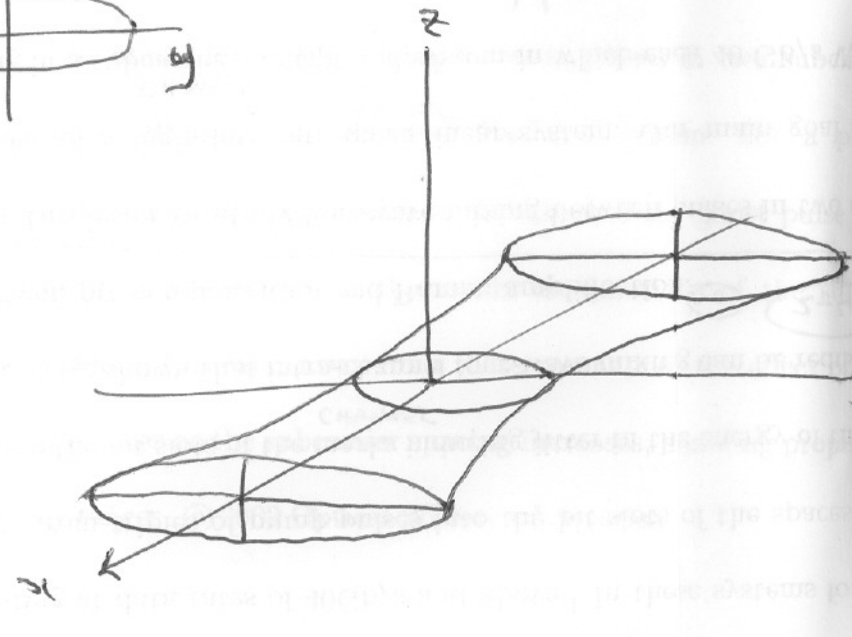
$\boxed{z=0}$ $y^2 - x^2 = 1$



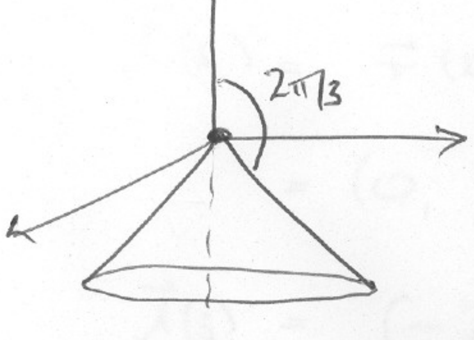
$\boxed{x=c}$ FOR ANY C

$y^2 + 4z^2 = 1 + c^2$





(b) $\phi = \frac{2\pi}{3} = 120°$



CONE. POINTING/OPENING TOWARDS −ve Z. AXIS

(3) [15 pts]
Consider the plane through $(0, 0, 0)$ with normal vector $(4, 2, 3)$.
(a) Find an equation of the form $ax + by + cz = d$ for this plane.

$\vec{x} = (x\ y\ z)$

$\vec{x}_0 = (0\ 0\ 0)$

$\vec{n} = (4\ 2\ 3)$

$(\vec{x} - \vec{x}_0) \cdot \vec{n} = 0$

$(x\ y\ z) \cdot (4\ 2\ 3) = 0$

$\boxed{4x + 2y + 3z = 0}$

(b) Find a parametrization of this plane.

By (a): $x = t$

$y = s$

$z = \dfrac{-4x - 2y}{3}$

$s, t \in \mathbb{R}$

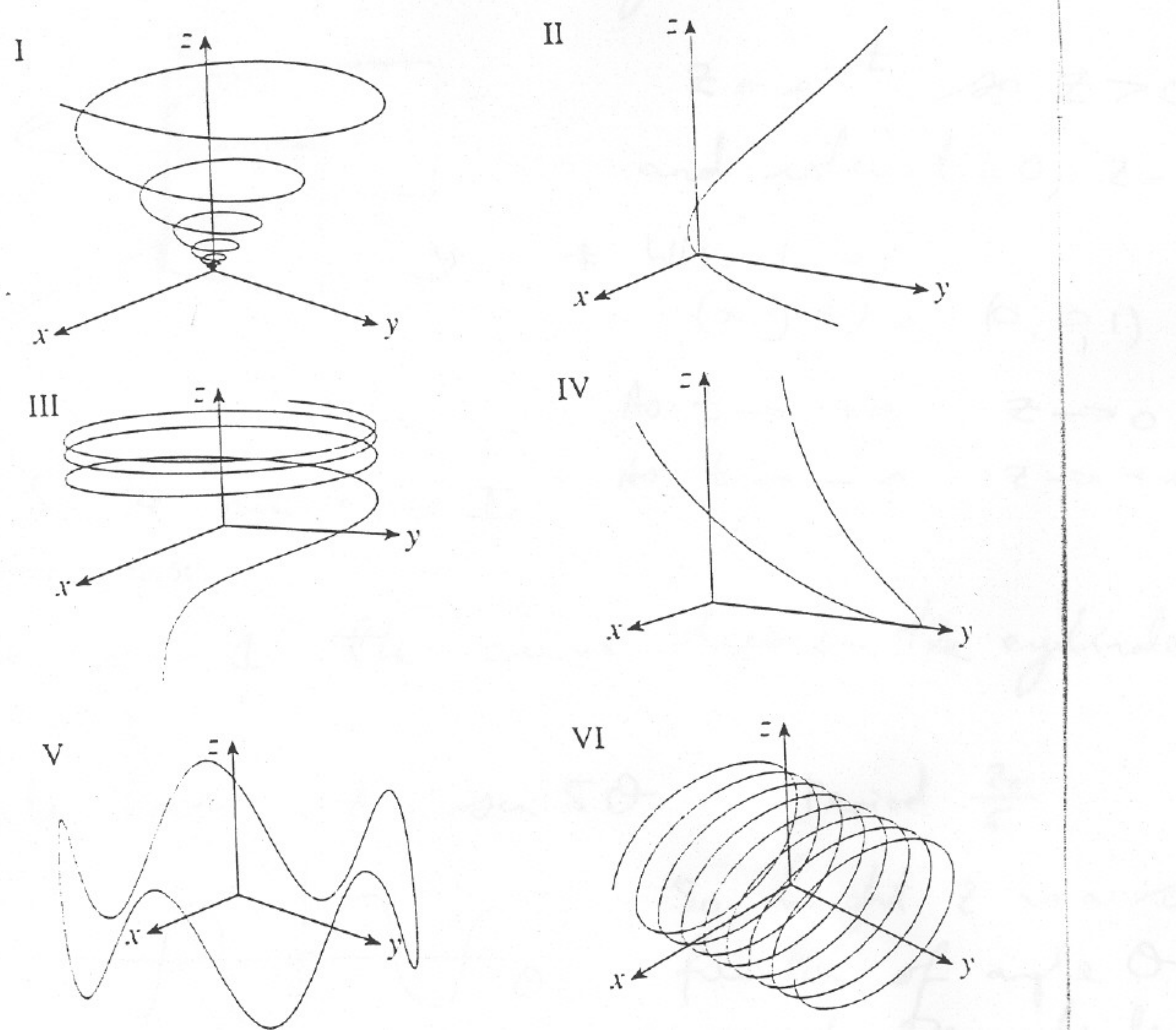
(c) Suppose that $\mathbf{r}(t)$ is a curve for which $\mathbf{r}(2) = (0, 1, -3)$, $\mathbf{r}'(2) = (-1, 2, 5)$, and $\mathbf{r}''(2) = (2, 4, -6)$. Find a parametrization of the tangent line to this curve at $t = 2$.

$\vec{l}(s) = \vec{r}(2) + s\ \vec{r}'(2)$

$= (0, 1, -3) + s(-1, 2, 5)$

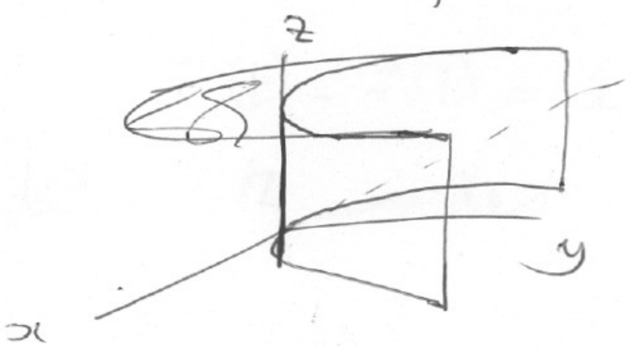
$\vec{l}(s) = (-s, 1 + 2s, -3 + 5s)$

(4) [15 pts] Match the parametric equations (a)-(b) on the next page with the graphs labeled (I)-(VI). [Note that there are more graphs than equations!] *Carefully explain the reasons for your choices.*



(a) $x = t$, $y = t^2$, $z = e^{-t}$.

The curve lies on the surface $y = x^2$
which is a parabolic cylinder



$z = e^{-t}$ so $z > 0$
and when $t = 0$, $z = 1$
* When $t = 0$
$(x\ y\ z) = (0, 0, 1)$

As $t \to +\infty$ $z \to 0$
As $t \to -\infty$ $z \to +\infty$.

So the curve is II.

(b) $r = 1$, $\theta = t$, $z = \sin 5t$.

Since $r = 1$ the curve lies on the cylinder
$x^2 + y^2 = 1$.
We also have $z = \sin 5\theta$ Period $\frac{2\pi}{5}$

So height z is a sine function of angle θ around the cylinder

Therefore answer is V.

(5) [10 pts] Suppose that $\mathbf{r}$ is a curve that lies on the sphere of radius 1 centered at the origin. Prove that at each point on the curve, the velocity vector $\mathbf{r}'(t)$ to the curve is perpendicular to the position vector $\mathbf{r}(t)$ of the point.

We know $|\vec{r}(t)| = 1$.

So $\vec{r}(t) \cdot \vec{r}(t) = 1$

Differentiate both sides with respect to t

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = 0$$

So $\vec{r}(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}(t) = 0$

by Product Rule

So $2\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0$

$\Rightarrow \vec{r}(t) \cdot \vec{r}\,'(t) = 0$

$\Rightarrow \vec{r}(t)$ is perpendicular to $\vec{r}\,'(t)$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________