NAME*:

1 /15	2 /15	3 /15	4 /25	5 /15	T /75

MATH 251 (Fall 2004) Exam 3, Nov 29th

No calculators, books or notes! Show all work and give **complete explanations** for all your answers. This 65 minute exam is worth 75 points.

(1) [15 pts]

(a) Suppose that $(0,2)$ is a critical point of a function g with continuous second partial derivatives. What can you say about g if

$$g_{xx}(0,2) = -1 \qquad g_{xy}(0,2) = 2 \qquad g_{yy}(0,2) = -8\,?$$

$$D = \det\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix} = 8 - 4 = 4 > 0$$

$g_{xx} < 0$ So by Second Derivative Test, g has a LOCAL MAX at $(0,2)$.

(b) Find the maximum rate of change of the function $f(x,y,z) = x^2y^3z^4$ at the point $(1,1,1)$, and the direction in which it occurs.

$$\nabla f = (2xy^3z^4,\ 3x^2y^2z^4,\ 4x^2y^3z^3)$$

$$\nabla f(1,1,1) = (2,3,4)$$

MAX RATE OF CHANGE IS $|\nabla f(1,1,1)| = \sqrt{4+9+16} = \sqrt{29}$

DIRN OF THIS RATE OF CHANGE IS

$$\vec{u} = \frac{\nabla f(1,1,1)}{|\nabla f(1,1,1)|} = \frac{1}{\sqrt{29}}(2,3,4)$$

(2) [15 pts]

(a) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = x^2y^3\mathbf{i} - y\sqrt{x}\mathbf{j}$, and where C is the curve parametrized by $\mathbf{r}(t) = t^2\mathbf{i} - t^3\mathbf{j}$ for $0 \le t \le 1$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

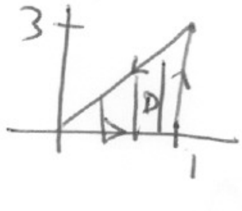
$$= \int_0^1 [t^4(-t^3)^3 \vec{i} - t^3 t \,\vec{j}] \cdot [2t\,\vec{i} - 3t^2 \vec{j}]\, dt$$

$$= \int_0^1 -t^{13}\cdot 2t + t^4\, 3t^2\, dt$$

$$= \int_0^1 -2t^{14} + 3t^6\, dt = \left[-\frac{2}{15}t^{15} + \frac{3}{7}t^7\right]_0^1$$

$$= -\frac{2}{15} + \frac{3}{7}$$

(b) Use Green's Theorem to evaluate $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$, where C is the curve that consists of straight lines joining $(0,0)$ to $(1,0)$, $(1,0)$ to $(1,3)$, and $(1,3)$ to $(0,0)$.



$C = \partial D$

$$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$$

$$= \int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (2y - 0)\, dA = 2\iint_D y\, dA$$

$$= 2\int_{x=0}^{x=1} \int_{y=0}^{y=3x} y\, dy\, dx$$

$$= \int_{x=0}^{x=1} [y^2]_{y=0}^{y=3x}\, dx = 9\int_0^1 x^2\, dx = 9\left[\frac{x^3}{3}\right]_0^1 = \underline{3}$$

(3) [15 pts]
Find the absolute maximum and minimum of the function $f(x, y) = xy$ on the region $3x^2 + 3y^2 \le 1$.

CRITICAL POINTS INSIDE REGION!

$(0,0) = \nabla f(x,y) = (y, x)$

$\Rightarrow (x,y) = (0,0) \qquad f(0,0) = 0.$

MAX/MIN f ON $3(x^2+y^2) = 1$

Use parametrization $\vec{r}(t) = \frac{1}{\sqrt{3}} \cos t\, \vec{i} + \frac{1}{\sqrt{3}} \sin t\, \vec{j}$ $\quad 0 \le t \le 2\pi$

[Then $3\left(\frac{1}{3}\cos^2 t + \frac{1}{3}\sin^2 t\right) = 1$ ✓]

Let $g(t) = f(\vec{r}(t)) = \frac{1}{3} \cos t \sin t$

$0 = g'(t) = -\frac{1}{3}\sin^2 t + \frac{1}{3}\cos^2 t$

So $\sin^2 t = \cos^2 t$ or $\tan^2 t = 1$

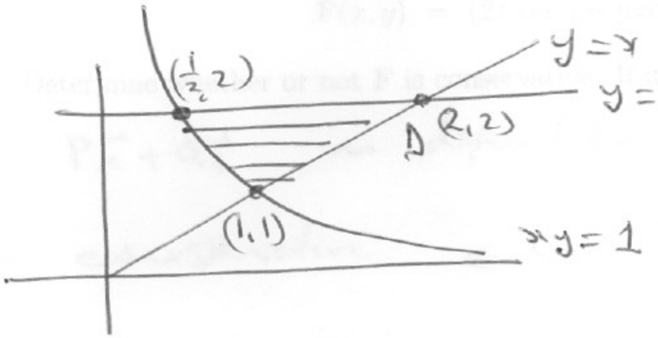
$\tan t = \pm 1 \qquad \boxed{t = \pm \pi/4} \ \boxed{\text{AND} \quad t = \pm \frac{3\pi}{4}}$

$g(\pm \pi/4) = \frac{1}{3} \cdot \frac{1}{\sqrt{2}}\left(\pm \frac{1}{\sqrt{2}}\right) = \pm \frac{1}{6} \qquad g\left(\pm \frac{3\pi}{4}\right) = \mp \frac{1}{6}$

ABS MAX IS $\frac{1}{6}$ AT $\vec{r}(\pi/4) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

ABS MIN IS $-\frac{1}{6}$ AT $\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

(4) [15 pts] Calculate the integral $\iint_D y\,dA$, where D is the region in the first quadrant that lies above the hyperbola $xy = 1$, above the line $y = x$, and below the line $y = 2$.



$1 \leq y \leq 2$

$\frac{1}{y} \leq x \leq y$

$$\iint_D y\,dA = \int_{y=1}^{y=2} \int_{x=\frac{1}{y}}^{x=y} y\,dx\,dy$$

$$= \int_{y=1}^{y=2} y\left(y - \frac{1}{y}\right) dy$$

$$= \int_1^2 y^2 - 1\,dy = \left[\frac{y^3}{3} - y\right]_1^2$$

$$= \frac{8}{3} - 2 - \frac{1}{3} + 1$$

$$= \frac{7}{3} - 1 = \frac{4}{3}$$

(5) [15 pts] Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = (2x\cos y - y\cos x)\mathbf{i} + (-x^2\sin y - \sin x)\mathbf{j}.$$

(a) Determine whether or not $\mathbf{F}$ is conservative. If it is, find a function f so that $\mathbf{F} = \nabla f$.

(i) $\vec{F} = P\vec{i} + Q\vec{j}$ is defined on all of $\mathbb{R}^2 = D$, open + simply connected

So $\vec{F}$ is conservative $\iff \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

Well $\frac{\partial Q}{\partial x} = -2x\sin y - \cos x = \frac{\partial P}{\partial y}$

(ii) $\frac{\partial f}{\partial x} = 2x\cos y - y\cos x \implies f(x,y) = x^2\cos y - y\sin x + g(y)$

$\frac{\partial f}{\partial y} = -x^2\sin y - \sin x \implies f(x,y) = x^2\cos y - y\sin x + g_2(x)$

So $\boxed{f(x,y) = x^2\cos y - y\sin x + C}$

(b) Let C be the curve that is the straight line from $(0,0)$ to $(1,1)$. What is $\int_C \mathbf{F}\cdot d\mathbf{r}$?

By FTC for LINE INTEGRALS

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(1,1) - f(0,0)$$

$$= 1\cos 1 - 1\sin 1 - (0-0)$$

$$= \cos 1 - \sin 1$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ____________________