NAME: SOLUTIONS

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MATH 251 (Spring 2004) Final Exam, May 12th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 120 minute exam. It is worth a total of 120 points.

(1) [12 pts]
(a) Let S be the surface $z = xy^2 + x^2e^{y-2}$. Find an equation of the form $ax + by + cz = d$ for the tangent plane to S at the point $(x_0, y_0, z_0) = (1, 2, 5)$.

$$z - z_0 = (x - x_0, y - y_0) \cdot \nabla f(x_0, y_0)$$

$$\nabla f = (y^2 + 2xe^{y-2}, 2xy + x^2e^{y-2})$$

$$\nabla f(1,2) = (4 + 2, 4 + 1) = (6, 5)$$

$$z - 5 = (6, 5) \cdot (x - 1, y - 2)$$

$$z = 5 + 6x - 6 + 5y - 10$$

$$\boxed{6x + 5y - z = 11}$$

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(b) Let C be the curve in space parametrized by $\mathbf{r}(t) = (t, t^2, t^3)$.
Find a parametrization of the tangent line to this curve at $t = 1$.

$$\vec{l}(s) = \vec{r}(1) + s\,\vec{r}\,'(1) \qquad \vec{r}\,'(t) = (1, 2t, 3t^2)$$

$$= (1, 1, 1) + s(1, 2, 3)$$

$$\vec{l}(s) = (1+s,\ 1+2s,\ 1+3s), \qquad s \in \mathbb{R}$$

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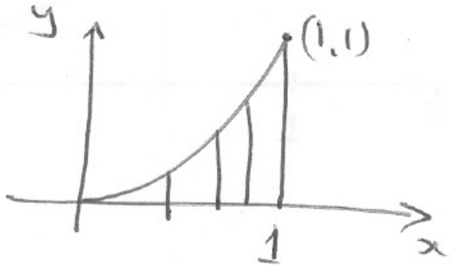
(2) [10 pts]
Let D be the region in the xy plane bounded by $y = 0$, $y = x^2$, and $x = 1$. Calculate $\iint_D x \cos y \, dA$



$0 \leq x \leq 1$

$0 \leq y \leq x^2$

$$\iint_D x \cos y \, dA$$

$$= \int_{x=0}^{x=1} x \int_{y=0}^{y=x^2} \cos y \, dy \, dx \qquad 6/10$$

$$= \int_{x=0}^{x=1} x \left[\sin y\right]_{y=0}^{y=x^2} dx$$

$$= \int_{x=0}^{x=1} x \sin x^2 \, dx \qquad u = x^2, \quad du = 2x\,dx$$

$$= \frac{1}{2}\int_{u=0}^{1} \sin u \, du = \frac{1}{2}\left[\cos u\right]_0^1 = \frac{1}{2}\left(1 - \cos(1)\right)$$

(3) [12 pts] Let f be the function $f(x, y) = x^2 + 3y^2 + 4xy$.
(a) In what direction is the rate of change of f the largest at the point $(x, y) = (2, 3)$, and what is the rate of change of f in this direction?

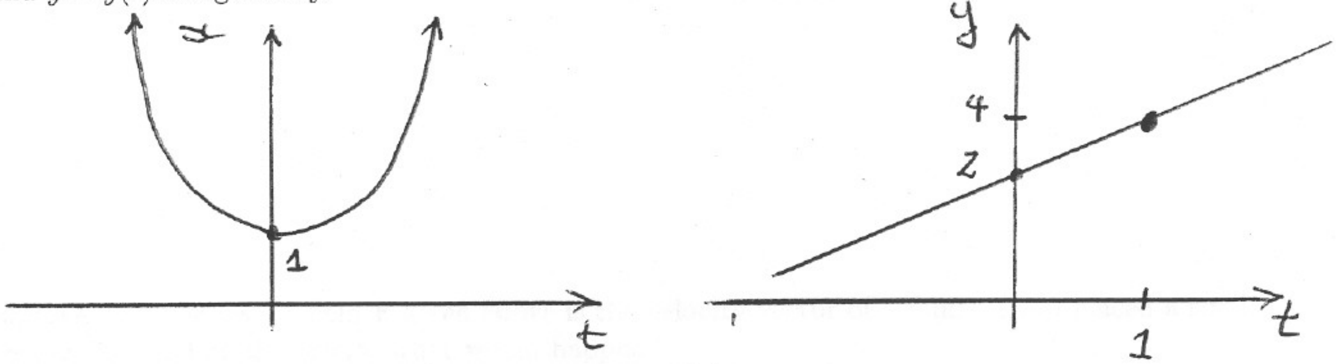
$$\nabla f(x,y) = (2x + 4y,\ 6y + 4x)$$

$$\nabla f(2,3) = (4 + 12,\ 18 + 8) = (16, 26)$$

DIRECTION $\vec{u} = \dfrac{\nabla f(2,3)}{|\nabla f(2,3)|} = \dfrac{(16, 26)}{\sqrt{16^2 + 26^2}}$ 3

Rate $= |\nabla f(2,3)| = \sqrt{16^2 + 26^2}$ 3

(b) Let $\mathbf{r}(t) = (x(t), y(t))$ be the parameterized curve in the plane for which the graphs of the functions $x = x(t)$ and $y = y(t)$ are given by:



Let $g = f \circ \mathbf{r}$, where the formula for f is given above. Find $g'(0)$.

$$g'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$ 3

$$= \nabla f(x(0), y(0)) \cdot (x'(0), y'(0))$$

$$= \nabla f(1, 2) \cdot (0, 2)$$

$$= (2 + 8,\ 12 + 4) \cdot (0, 2) = (10, 16) \cdot (0, 2)$$

$$= \underline{32}$$ 3

(4) [10 pts]
Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = x^2\mathbf{i} + yz\mathbf{j} + xe^{3z}\mathbf{k}$.
(a) Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

$$\nabla \cdot \vec{F} = 2x + z + 3xe^{3z}$$

3

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & yz & xe^{3z} \end{vmatrix} = (-y, -e^{3z}, 0)$$

3

(b) Now suppose that the vector field $\mathbf{F}$ given above is the velocity vector of a fluid. If you placed a small paddle wheel in the fluid at the origin, what would happen?

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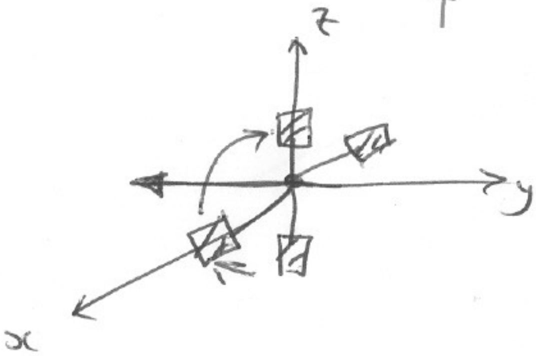
$$\nabla \times \vec{F}(0,0,0) = (0, -1, 0)$$



Its axis would align with negative y axis and rotate about this axis. with dirn given by RH rule

(5) [18 pts]
(a) Set up an iterated triple integral that equals the volume of the solid that is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $2y + 3z = 6$, and the surface $x = yz$. Do NOT attempt to evaluate the triple integral.

$0 \le y \le 3$

$0 \le z \le \frac{6-2y}{3}$

$0 \le x \le yz$

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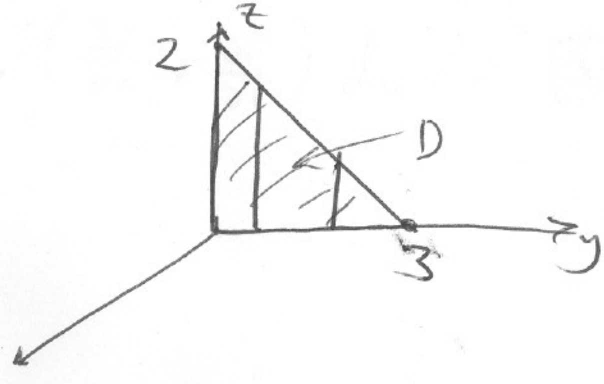


VOLUME

$$= \int_{y=0}^{y=3} \int_{z=0}^{z=\frac{6-2y}{3}} \int_{x=0}^{x=yz} 1 \, dx \, dz \, dy$$

3 ~~6~~

8

(b) Compute $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$, and S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with an upward normal.

$$\int_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) du\, dv \quad \boxed{3}$$

~~$\vec{r}(u,v) = u, v,$~~

$$\vec{r}(u,v) = (u\cos v,\ u\sin v,\ u) \qquad 0 \le v \le 2\pi$$

$$0 \le u \le 1$$

$\boxed{2}$

$$\frac{\partial \vec{r}}{\partial u} = (\cos v,\ \sin v,\ 1)$$

$$\frac{\partial \vec{r}}{\partial v} = (-u\sin v,\ u\cos v,\ 0)$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-u\cos v, -u\sin v, u) \quad \boxed{2}$$

When $u = 1$, $v = 0$; $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (-1, 0, 1)$ $\boxed{1}$ is upward

$+$

$$\int_S \vec{F} \cdot d\vec{S} = \int_{u=0}^{1} \int_{v=0}^{2\pi} (u\cos v, u\sin v, u^4) \cdot (-u\cos v, -u\sin v, u)\, dv\, du$$

$$= \int_{u=0}^{1} \int_{v=0}^{2\pi} -u^2 + u^5\, dv\, du = 2\pi\left[-\frac{1}{3} + \frac{1}{6}\right] \boxed{2} = \boxed{\cancel{\ }}$$

$-\pi/3$

10

(6) [15 pts] Find the local maximum and minimum values and saddle points of the function

$$f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

$$\nabla f = (6x^2 + y^2 + 10x,\ 2xy + 2y) = (0,0)$$

at $2(x+1)y = 0 \Rightarrow y = 0$ or $x = -1$

$\boxed{x=-1}$ $6 + y^2 - 10 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$ $(-1, \pm 2)$

$\boxed{y=0}$ $0 = 6x^2 + 10x = 2x(3x+5) \Rightarrow x = 0$ $(0,0)$

or $x = -5/3$ $(-5/3, 0)$

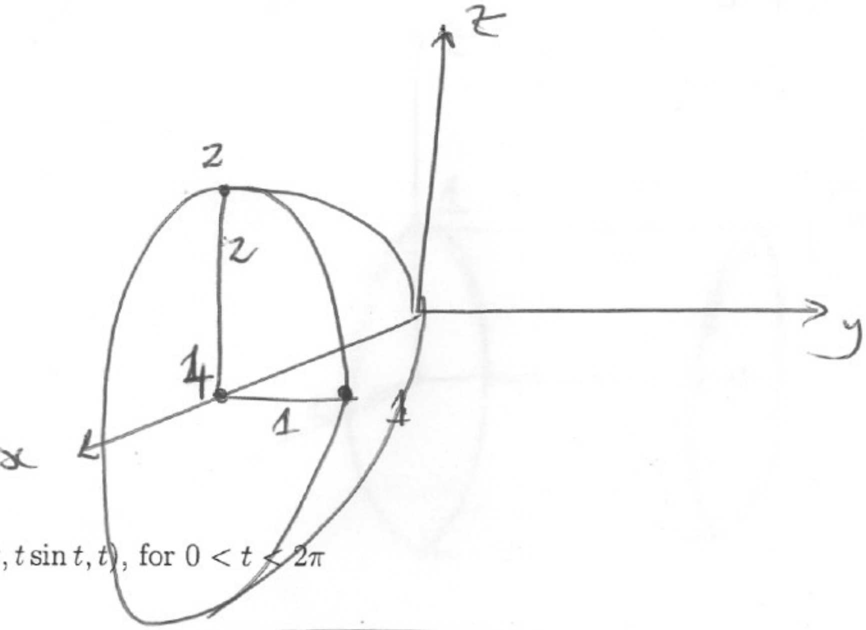
$$D = \det\begin{pmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{pmatrix} = 4[(x+1)(6x+5) - y^2]$$

	D	f_{xx}	
$(-1,2)$	$-$		Saddle Pt
$(-1,-2)$	$-$		Saddle Pt
$(0,0)$	$+$	$+$	Min
$(-5/3,0)$	$+$	$-$	Max

(7) [16 pts] Sketch the following curves and surfaces

(a) $x = 4y^2 + z^2$

Core MAx 4



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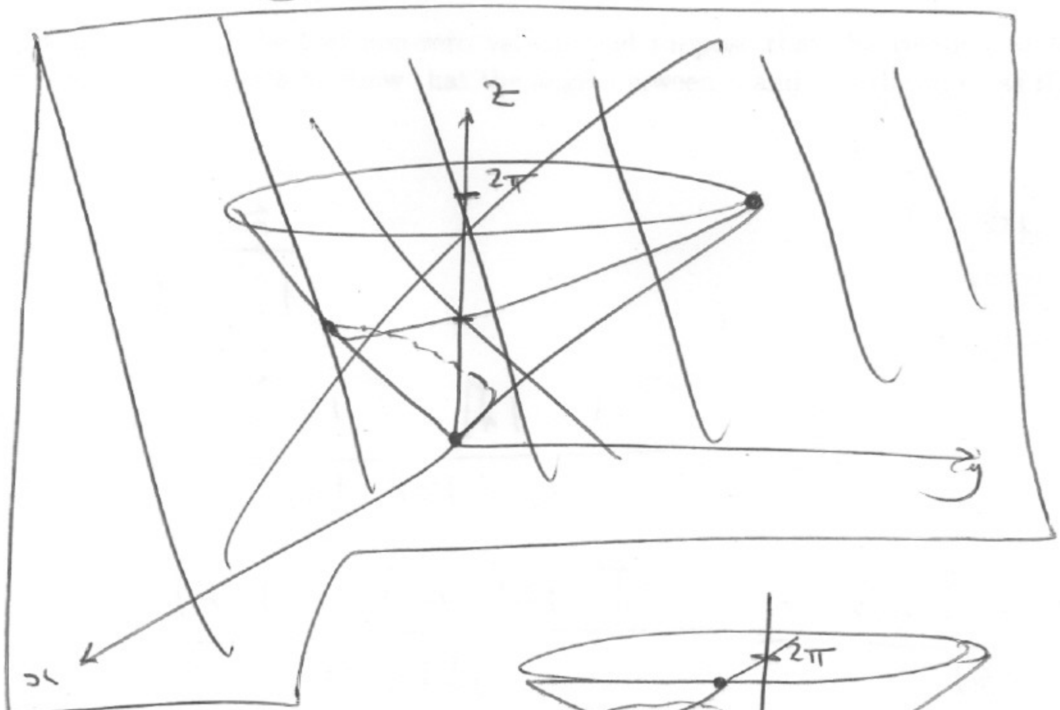
(c) $\mathbf{r}(t) = (t\cos t, t\sin t, t)$, for $0 < t < 2\pi$



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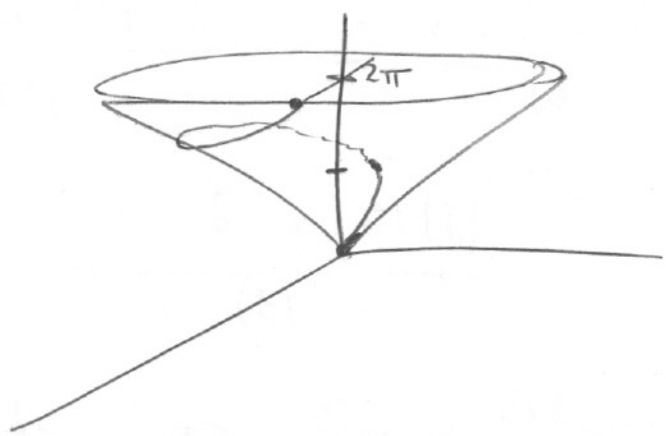
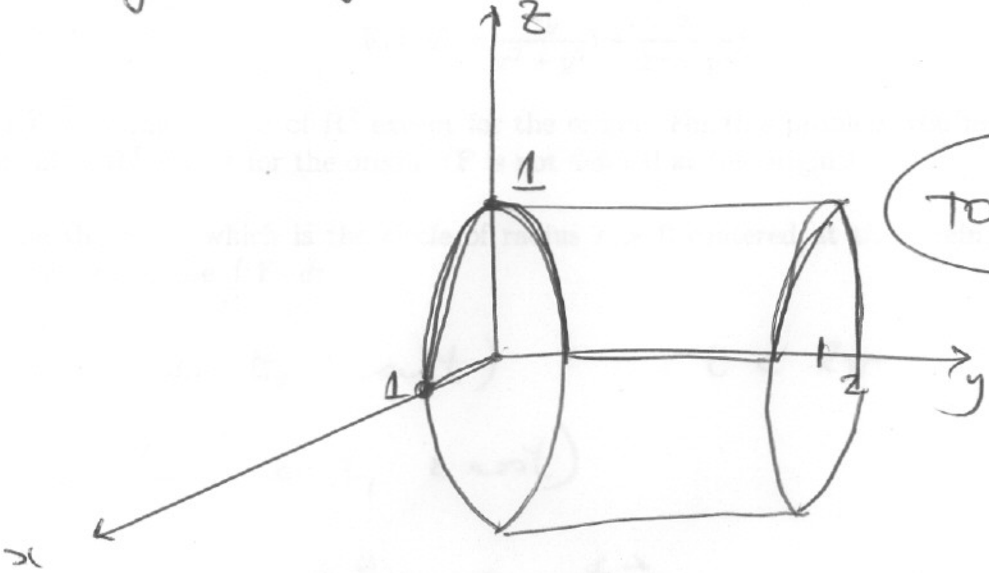


(d) $\mathbf{r}(x,\theta) = (\cos\theta, \cancel{x}\, y, \sin\theta)$ for $0 < \cancel{x}\, y < 2$ and $0 < \theta < \pi$

$x^2 + z^2 = 1$



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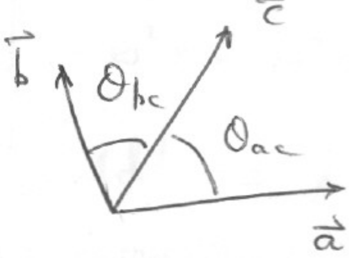
(8) [10 pts] Let **a** and **b** be two non-zero vectors and suppose that the vector $\mathbf{c} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}$ is also non-zero. Use vector algebra to show that the angle between **a** and **c** is the same as the angle between **b** and **c**.



$$\cos\theta_{ac} = \frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\,|\vec{c}|}$$

$$= \frac{\vec{a}\cdot\left(|\vec{a}|\,\vec{b} + |\vec{b}|\,\vec{a}\right)}{|\vec{a}|\,|\vec{c}|}$$

$$= \frac{|\vec{a}|\,\vec{a}\cdot\vec{b} + |\vec{a}|^2\,|\vec{b}|}{|\vec{a}|\,|\vec{c}|} = \frac{\vec{a}\cdot\vec{b} + |\vec{a}|\,|\vec{b}|}{|\vec{c}|}$$

$$\cos\theta_{bc} = \frac{\vec{b}\cdot\vec{c}}{|\vec{b}|\,|\vec{c}|} = \frac{\vec{a}\cdot\vec{b} + |\vec{a}|\,|\vec{b}|}{|\vec{c}|}$$

So $\cos\theta_{ac} = \cos\theta_{bc} \Rightarrow \theta_{ac} = \theta_{bc}$ as both angles are in $[0,\pi]$.

(9) [12 pts] Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be the vector field given by

$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

Notice that $\mathbf{F}$ is defined on all of $\mathbf{R}^2$ except for the origin. For this problem you may use the fact that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on all of $\mathbf{R}^2$ except for the origin. ($\mathbf{F}$ is not defined at the origin.)

(a) Let C_ϵ be the curve which is the circle of radius $\epsilon > 0$ centered at the origin, with counterclockwise orientation. Compute $\int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{r}$.

$\vec{r}(t) = (\varepsilon \cos t, \varepsilon \sin t) \quad 0 < t < 2\pi$

$\vec{r}\,'(t) = (-\varepsilon \sin t, \varepsilon \cos t)$

$\vec{F}(\vec{r}(t)) = \dfrac{-\varepsilon \sin t\, \vec{i} + \varepsilon \cos t\, \vec{j}}{\varepsilon^2}$

So $\displaystyle\int_{C_\varepsilon} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \cos^2 t + \sin^2 t\,dt$

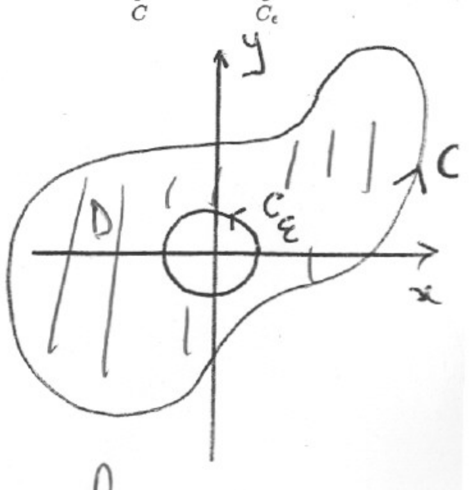
$\displaystyle = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$

(b) Let C be the oriented curve shown in the diagram. Use Green's Theorem to show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{r}$.

Let D be the shaded region

between C_ε and C.

$\partial D = C - C_\varepsilon$



So

$\displaystyle 0 = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{=\,0 \text{ AS } (0,0) \notin D} = \int_C \vec{F} \cdot d\vec{r} - \int_{C_\varepsilon} \vec{F} \cdot d\vec{r}$

So $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_{C_\varepsilon} \vec{F} \cdot d\vec{r}$

(10) [5 pts] Let $\mathbf{F}$ be a vector field on $\mathbf{R}^3$ whose component functions have continuous partial derivatives. Calculate the value of $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where the surface S is the sphere of radius 1 centered at the origin, with outward normal vector.

$\partial S = \emptyset$ is EMPTY

Stokes' Thm says

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r} = 0$$

as ∂S is empty.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________