MATH 251 (Spring 2004) Final Exam, May 12th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 120 minute exam. It is worth a total of 120 points.

(1) [12 pts]
(a) Let S be the surface $z = xy^2 + x^2 e^{y-2}$. Find an equation of the form $ax + by + cz = d$ for the tangent plane to S at the point $(x_0, y_0, z_0) = (1, 2, 5)$.

(b) Let C be the curve in space parametrized by $\mathbf{r}(t) = (t, t^2, t^3)$.
Find a parametrization of the tangent line to this curve at $t = 1$.

(2) [10 pts]
Let D be the region in the xy plane bounded by $y = 0$, $y = x^2$, and $x = 1$. Calculate $\iint_D x \cos y \, dA$

(3) [12 pts] Let f be the function $f(x, y) = x^2 + 3y^2 + 4xy$.
(a) In what direction is the rate of change of f the largest at the point $(x, y) = (2, 3)$, and what is the rate of change of f in this direction?

(b) Let $\mathbf{r}(t) = (x(t), y(t))$ be the parameterized curve in the plane for which the graphs of the functions $x = x(t)$ and $y = y(t)$ are given by:

Let $g = f \circ \mathbf{r}$, where the formula for f is given above. Find $g'(0)$.

(4) [10 pts]
Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = x^2\mathbf{i} + yz\mathbf{j} + xe^{3z}\mathbf{k}$.
(a) Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

(b) Now suppose that the vector field $\mathbf{F}$ given above is the velocity vector of a fluid. If you placed a small paddle wheel in the fluid at the origin, what would happen?

(5) [18 pts]

(a) Set up an iterated triple integral that equals the volume of the solid that is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $2y + 3z = 6$, and the surface $x = yz$. Do NOT attempt to evaluate the triple integral.

(b) Compute $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$, and S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with an upward normal.

(6) [15 pts] Find the local maximum and minimum values and saddle points of the function

$$f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

(7) [16 pts] Sketch the following curves and surfaces
(a) $x = 4y^2 + z^2$

(c) $\mathbf{r}(t) = (t\cos t, t\sin t, t)$, for $0 < t < 2\pi$

(d) $\mathbf{r}(x, \theta) = (\cos\theta, x, \sin\theta)$ for $0 < x < 2$ and $0 < \theta < \pi$

(8) [10 pts] Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors and suppose that the vector $\mathbf{c} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}$ is also non-zero. Use vector algebra to show that the angle between $\mathbf{a}$ and $\mathbf{c}$ is the same as the angle between $\mathbf{b}$ and $\mathbf{c}$.

(9) [12 pts] Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be the vector field given by

$$\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

Notice that $\mathbf{F}$ is defined on all of $\mathbf{R}^2$ except for the origin. For this problem you may use the fact that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on all of $\mathbf{R}^2$ except for the origin. ($\mathbf{F}$ is not defined at the origin.)

(a) Let C_ϵ be the curve which is the circle of radius $\epsilon > 0$ centered at the origin, with counterclockwise orientation. Compute $\int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{r}$.

(b) Let C be the oriented curve shown in the diagram. Use Green's Theorem to show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_\epsilon} \mathbf{F} \cdot d\mathbf{r}$.

(10) [5 pts] Let $\mathbf{F}$ be a vector field on $\mathbf{R}^3$ whose component functions have continuous partial derivatives. Calculate the value of $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where the surface S is the sphere of radius 1 centered at the origin, with outward normal vector.

Pledge: *I have neither given nor received aid on this exam*

Signature: ________________________________