NAME: SOLUTIONS

1 /10	2 /12	3 /12	4 /10	5 /6	T /50

MATH 251 (Fall 2009) Exam I, Sept 28th

No calculators, books or notes! Show all work and give **complete explanations**. This is 65 min exam is worth 50 points.

(1) [10 pts]

(a) Calculate the projection, $\mathrm{Proj}_{\mathbf{v}}(\mathbf{w})$, of the vector $\mathbf{w} = (1, -2, 5)$ onto the vector $\mathbf{v} = (0, 4, -3)$.

$$\mathrm{PROJ}_{\vec{v}}(\vec{w}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|^2}\vec{v} \quad (\text{Parallel to } \vec{v}\ !!)$$

$$= \frac{(0,4,-3)\cdot(1,-2,5)}{\sqrt{0^2+4^2+(-3)^2}^{\,2}}(0,4,-3) = \frac{-23}{25}(0,4,-3)$$

(b) Calculate the volume of the parallelipiped with three adjacent edges given by the vectors $\mathbf{a} = (2, 1, 0)$, $\mathbf{b} = (1, 3, 0)$, and $\mathbf{c} = (1, 2, -4)$.

$$\mathrm{VOL} = |(\vec{a}\times\vec{b})\cdot\vec{c}| = \left|\det\begin{bmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{bmatrix}\right|$$

$$\mathrm{VOL} = \left|\begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & -4 \end{vmatrix}\right|$$

$$= \left| 2\begin{vmatrix} 3 & 0 \\ 2 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & -4 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\right|$$

$$= |-24 + 4 + 0| = |-20| = 20.$$

(2) [12 pts]

(a) Find a vector parametric equation for the line through the point $(1, 2, -1)$ that is normal to the plane $2x - y + 3z = 12$.

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

where $\vec{r}_0 = (1, 2, -1)$ is point on line

$\vec{v}$ = vector in direction of line

= normal to plane = $(2, -1, 3)$

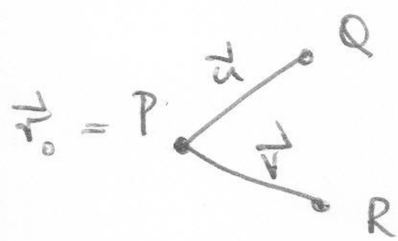
which read off from coefficients of $2x - y + 3z = 12$.

So $\vec{r}(t) = (1, 2, -1) + t(2, -1, 3)$

$= (1 + 2t,\ 2 - t,\ -1 + 3t)$

(b) Find a parametrization of the plane containing the point $(1, -2, 1)$, $(2, -1, 0)$ and $(3, -2, 2)$.

P Q R



$$\vec{r}_0 = P = (1, -2, 1)$$

$$\vec{u} = \overrightarrow{PQ} = Q - P = (1, 1, -1)$$

$$\vec{v} = \overrightarrow{PR} = R - P = (2, 0, 1)$$

$$\vec{r}(s, t) = \vec{r}_0 + s\vec{u} + t\vec{v}$$

$$= (1, -2, 1) + s(1, 1, -1) + t(2, 0, 1)$$

$$= (1 + s + 2t,\ -2 + s,\ 1 - s + t)$$

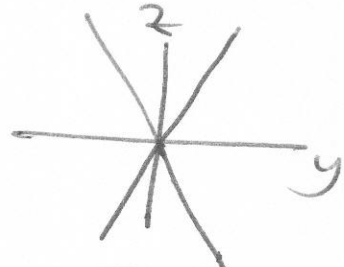
(3) [12 pts] Consider the quadric surface

$$z^2 = x^2 + 4y^2.$$

Find the equations for the slices (i.e., traces) of this surface in the planes $x = k$, $y = k$, $z = k$ for a few appropriately chosen values of k. Sketch each of these traces in a plane. Then sketch the surface in space.

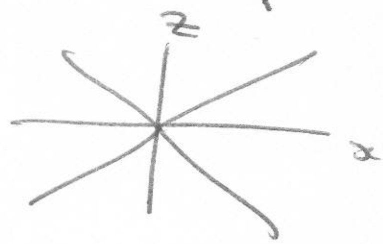
$\underline{x = 0}$ $\quad z^2 = 4y^2$

$z = \pm 2y$



$\underline{y = 0}$ $\quad z^2 = x^2$

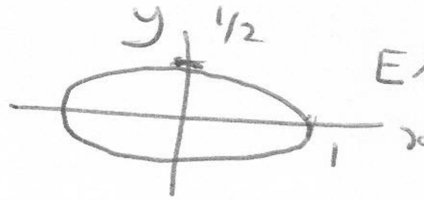
$z = \pm x$



$\underline{z = 0}$ $\quad x^2 + 4y^2 = 0$

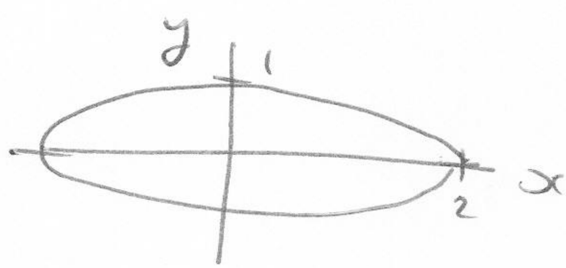
ORIGIN

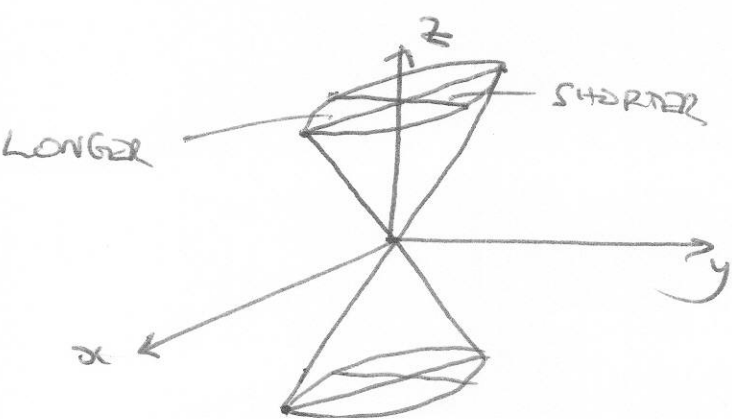
$\underline{z = \pm 1}$ $\quad x^2 + 4y^2 = 1$



Ellipse

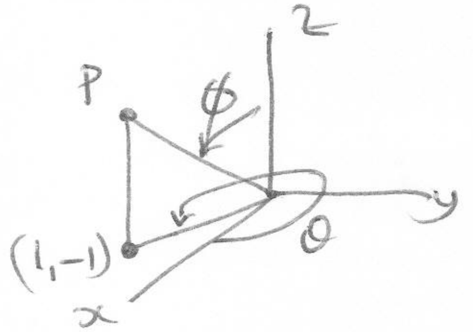
$\underline{z = \pm 2}$ $\quad x^2 + 4y^2 = 4$





Double Elliptical Cone

(4) [10 pts]
(a) Convert the point $(x, y, z) = (1, -1, 1)$ in rectangular coordinates to spherical coordinates. [Hint: You may find it helpful to draw a picture.]



$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{3}$$

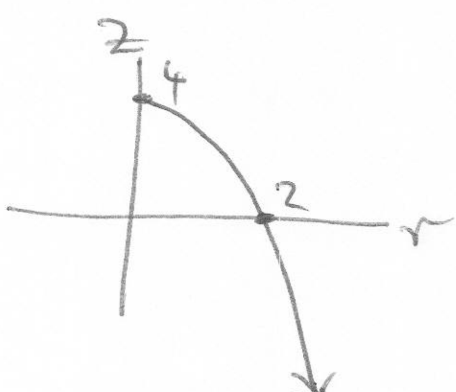
$$\theta = \frac{7\pi}{4}$$

$$z = \rho \cos\phi$$

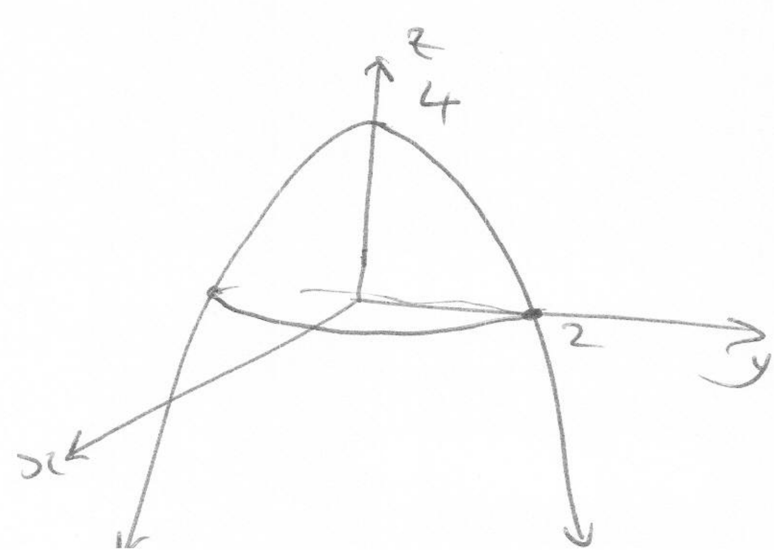
$$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{1}{\sqrt{3}}\right)$$

(ϕ is in range $[0, \pi]$ as required)

(b) Sketch the graph of the surface whose equation is cylindrical coordinates is $z = 4 - r^2$.



As eqn is θ independent ~~spin~~
SPIN around z axis to get



Upside Down
Translated
Circular
Paraboloid

(5) [6 pts] Which of the following statements are *always true* and which are *not always true*. Give reasons for your answers.

(a) $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$

$\vec{\imath} \times \vec{\jmath} = \vec{k}$

$\vec{\jmath} \times \vec{\imath} = -\vec{k}$

(we know $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$
$= |\vec{v}||\vec{u}| \sin\theta$
$= |\vec{v} \times \vec{u}|$)

So NOT ALWAYS TRUE

In fact by Right Hand Rule $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

(b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$

$\vec{u} \times \vec{v} \perp \vec{u}$. $\Rightarrow \theta = \pi/2$

So $(\vec{u} \times \vec{v}) \cdot \vec{u} = |\vec{u} \times \vec{v}||\vec{u}| \cos\theta$

$= 0.$

ALWAYS TRUE

(c) $\mathbf{u} \times \mathbf{u} = |\mathbf{u}|^2$

NEVER TRUE

$\vec{u} \times \vec{u}$ is a vector (it is $\vec{0}$)

BUT $|u|^2$ is a scalar (it can be $\neq 0$)

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________