NAME: SOLUTIONS

1 /8	2 /9	3 /9	4 /12	5 /12	T /50

MATH 251 (Fall 2009) Exam II, Oct 30th

No calculators, books or notes! Show all work and give **complete explanations**. This is 65 min exam is worth 50 points.

(1) [8 pts] Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be the parametrized curve

$$\mathbf{r}(t) = (t^2, e^{3t}, \cos(4t))$$

and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function such that

$$f(0,1,1) = 5 \qquad f(0,3,0) = -2$$
$$\nabla f(0,1,1) = 2\mathbf{i} - 5\mathbf{j} + 7\mathbf{k} \qquad \nabla f(0,3,0) = -\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}.$$

Let $g(t) = f(\mathbf{r}(t))$. Find $g'(0)$.

$$g'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

$$\vec{r}(0) = (0, 1, 1)$$

$$\vec{r}\,'(t) = (2t, 3e^{3t}, -4\sin 4t)$$

$$\vec{r}\,'(0) = (0, 3, 0)$$

So

$$g'(0) = \nabla f(0,1,1) \cdot (0,3,0)$$

$$= (2,-5,7) \cdot (0,3,0)$$

$$= -15$$

(2) [9 pts]
(a) Set up *but do not evaluate* an integral to calculate the length of the parametrized curve

$$\mathbf{r}(t) = (t^2, e^{3t}, \cos(4t)), \qquad 0 \le t \le \pi.$$

That is, find numbers a and b and a function F so that the length of the curve is given by $\int_a^b F(t)\,dt$.

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$\vec{r}\,'(t) = (2t, 3e^{3t}, -4\sin 4t)$$

$$|\vec{r}\,'(t)|^2 = 4t^2 + 9e^{6t} + 16\sin^2 4t$$

$$L = \int_0^{\pi} \sqrt{4t^2 + 9e^{6t} + 16\sin^2 4t}\,dt$$

(b) Calculate the curvature of the parametrized curve $\mathbf{r}(t) = (3+2t, 5-t^2)$ at $t = 0$.

$$r'(t) = (2, -2t)$$

$$|\vec{r}\,'(t)| = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left((1+t^2)^{-1/2}, -t(1+t^2)^{-1/2}\right)$$

$$\vec{T}\,'(t) = \left(-\tfrac{1}{2}(1+t^2)^{-3/2}\cdot 2t, -(1+t^2)^{-1/2} - t\left(-\tfrac{1}{2}\right)(1+t^2)^{-3/2}2t\right)$$

$$= \left(-t(1+t^2)^{-3/2}, -(1+t^2)^{-1/2} + t^2(1+t^2)^{-3/2}\right)$$

$$K(t) = \frac{|T'(t)|}{|\vec{r}\,'(t)|} \qquad K(0) = \frac{|T'(0)|}{|\vec{r}\,'(0)|} = \frac{|(0,1)|}{2} = \frac{1}{2}$$

(3) [9 pts] Let $z = f(x, y)$ be a function with table of values given by

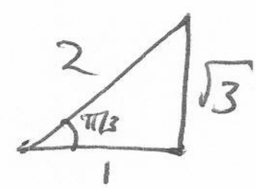
		y		
		4	5	6
x	0	7	8	5
	1	6	9	12
	2	8	11	15

Estimate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(x, y) = (1, 5)$. Use your answer to estimate the directional derivative of f in the direction $\theta = \pi/3$ at the point $(1, 5)$.

$$\frac{\partial f}{\partial x} \simeq \frac{f(2,5) - f(1,5)}{1} = \frac{11-9}{1} = 2$$

$$\frac{\partial f}{\partial y} \simeq \frac{f(1,6) - f(1,5)}{1} = \frac{12-9}{1} = 3.$$

$$\vec{v} = (\cos \pi/3, \sin \pi/3) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$



$$D_{\vec{v}} f = \nabla f \cdot \vec{v} \simeq (2,3) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 1 + \frac{3\sqrt{3}}{2}.$$

(4) [12 pts]
(a) Sketch and describe the surface with parametrization

$$x = r\cos\theta, \qquad y = 1 - r(\cos\theta + 2\sin\theta), \qquad z = r\sin\theta$$

where $0 \le \theta \le 2\pi$ and $0 \le r \le 3$.

$y = 1 - x - 2z$

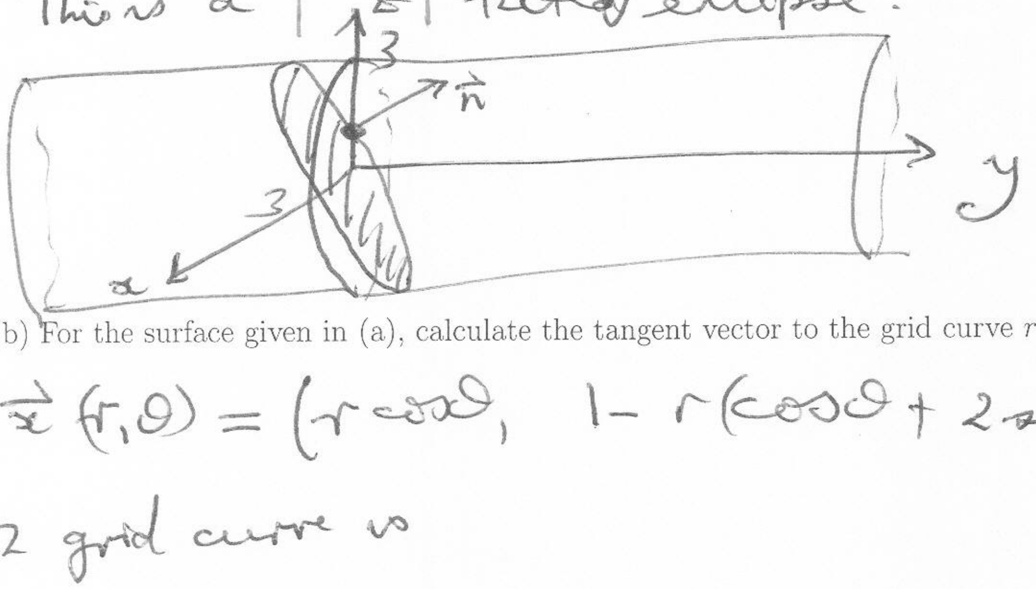
$x + y + 2z = 1$ Plane

Requiring $0 \le r \le 3$ means $x^2 + z^2 = r^2 \le 9$

So we are looking at part of plane

That lies inside cylinder $x^2 + z^2 = 9$

This is a tilted ellipse.



$(0, 0, \frac{1}{2})$

lies on plane

$\vec{n} = (1, 1, 2)$

(b) For the surface given in (a), calculate the tangent vector to the grid curve $r = 2$ when $\theta = \pi/4$.

$\vec{x}(r, \theta) = (r\cos\theta,\ 1 - r(\cos\theta + 2\sin\theta),\ r\sin\theta)$

$r = 2$ grid curve is

$\vec{x}(2, \theta) = (2\cos\theta,\ 1 - 2\cos\theta - 4\sin\theta,\ 2\sin\theta)$

$\frac{\partial \vec{x}}{\partial \theta}(2, \theta) = (-2\sin\theta,\ 2\sin\theta - 4\cos\theta,\ 2\cos\theta)$

$\frac{\partial \vec{x}}{\partial \theta}(2, \pi/4) = \left(\frac{-2}{\sqrt{2}},\ \frac{2}{\sqrt{2}} - \frac{4}{\sqrt{2}},\ \frac{2}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}}(-1, -1, 1)$

(5) [12 pts] Find the absolute maximum and minimum of the function $z = f(x, y) = (x + 1)^2 + y^2$ on the domain $x^2 + 4y^2 \leq 4$.

① Find value of f at critical pts inside domain D given by $x^2+4y^2 \leq 4$.

$\nabla f = (2(x+1), 2y) = (0,0)$ at $(x,y) = (-1,0)$

Since $(-1)^2 + 4 \cdot 0^2 = 1 \leq 4$ $(-1,0)$ is in D.

$\boxed{f(-1,0) = 0.}$

② Find abs max+min of f on $x^2+4y^2=4$

OPTION 1 Lagrange Multipliers

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases}$$

$2(x+1) = \lambda 2x$ ①

$2y = \lambda 8y$ ②

$x^2+4y^2=4$ ③

By ② $y(1-4\lambda)=0$

So $y=0$ OR $\lambda = \frac{1}{4}$.

PTO

Pledge: *I have neither given nor received aid on this exam*

Signature: ____________________

$\boxed{y=0}$ By ③ $x = \pm 2$

By ① $\lambda = \frac{x+1}{x} = \frac{\pm 2+1}{\pm 2} = \frac{3}{2}$ or $\frac{1}{2}$

So get $(x, y, \lambda) = (2, 0, \frac{3}{2}), (-2, 0, \frac{1}{2})$

$\boxed{f(2,0) = 9}$ $\boxed{f(-2,0) = 1}$

OR $\boxed{\lambda = \frac{1}{4}}$ By ① $x+1 = \frac{1}{4}x$

$4x+4 = x$

$3x = -4$

$x = -4/3$

By ③ $y^2 = 1 - \frac{1}{4}x^2 = 1 - \frac{1}{4}\frac{16}{9} = 1 - \frac{4}{9} = \frac{5}{9}$

$y = \pm \frac{\sqrt{5}}{3}$

So $(x, y, \lambda) = \left(-\frac{4}{3}, \pm\frac{\sqrt{5}}{3}, \frac{1}{4}\right)$

$\boxed{f\left(\frac{-4}{3}, \pm\frac{\sqrt{5}}{3}\right) =}$ $\frac{1}{9} + \frac{5}{9} = \frac{6}{9} = \boxed{2/3}$

ABS MAX $f(2,0) = 9$

ABS MIN $f(\cancel{1}, 0) =$

OPTION 2

Parametrize $x^2+4y^2=4$ using

$x = 2\cos t$

$y = \sin t$

Plug into $f(x,y) = (x+1)^2 + y^2$

$g(t) = f(2\cos t, \sin t) = (2\cos t + 1)^2 + \sin^2 t$

$= 4\cos^2 t + 4\cos t + 1 + \sin^2 t$

$g(t) = 3\cos^2 t + 4\cos t + 2$

$g'(t) = -6\cos t \sin t - 4\sin t$

$= \cancel{-2(3} \quad -2\sin t(2 + 3\cos t)$

So $\sin t = 0$ or $\cos t = -2/3$

ie $y = 0$ or $x = -4/3$

If $y=0$ then $x = \pm 2$ on ellipse

$f(\pm 2, 0) = \begin{cases} 9 \\ 1 \end{cases}$

If $x = -4/3$ then $y = \pm\frac{\sqrt{5}}{3}$ on ellipse

$f(-4/3, \pm\sqrt{5}/3) = 2/3$ So as before $f(-1,0) = 0$

Abs Max $f(2,0) = 9$