NAME: SOLUTIONS

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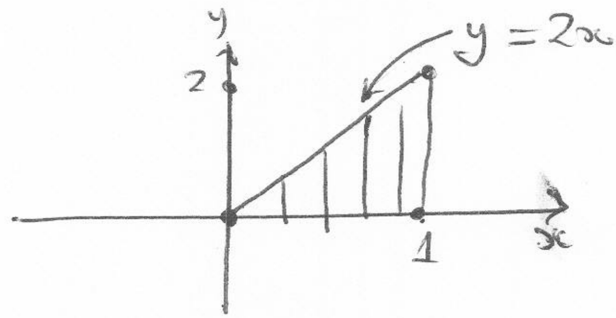
MATH 251 (Fall 2009) Exam III, Nov 25th

No calculators, books or notes! Show all work and give **complete explanations**. This is 65 min exam is worth 50 points.

(1) [10 pts] Calculate $\iint_D x\,dA$, where D is the triangle in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(1,2)$.

D is described by



$$0 \le x \le 1$$

$$0 \le y \le 2x$$

So

$$\iint_D x\,dA = \int_{x=0}^{x=1}\int_{y=0}^{y=2x} x\,dy\,dx$$

$$= \int_{x=0}^{x=1} x \cdot \int_{y=0}^{y=2x} 1\,dy\,dx$$

$$= \int_{x=0}^{x=1} x\,\big[y\big]_{y=0}^{y=2x}\,dx$$

$$= \int_0^1 x \cdot 2x\,dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}.$$

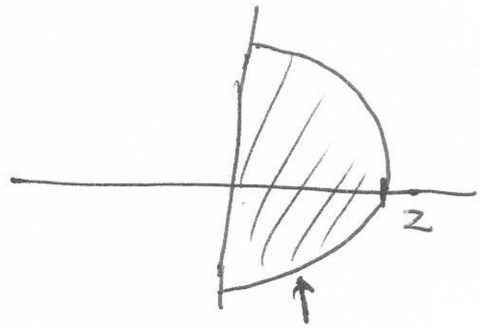
(2) [10 pts] Evaluate the integral

$$I = \int_{x=0}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=+\sqrt{4-x^2}} x \, dy \, dx,$$

by converting it to polar coordinates.

$$0 \le x \le 2$$

$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$



So in polar coordinates region is

$$-\frac{\pi}{2} \le \theta \le \pi/2$$

$$0 \le r \le 2$$

So

$$I = \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=2} (r\cos\theta)\, r \, dr \, d\theta$$

$$= \left(\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \right) \left(\int_0^2 r^2 \, dr \right)$$

$$= \left[\sin\theta \right]_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \right]_0^2$$

$$= (1-(-1)) \frac{8}{3} = \frac{16}{3}$$

(3) [10 pts] Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = xy\mathbf{i} + 3z\mathbf{j} + y\mathbf{k}$ and let C be the curve parametrized by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$, where $0 \le t \le 1$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\vec{r}\,'(t) = \vec{i} + 2t\vec{j} + 0\vec{k}$$

$$\vec{F}(\vec{r}(t)) = t\,t^2\vec{i} + 3.1\vec{j} + t^2\vec{k}$$

$$= t^3\vec{i} + 3\vec{j} + t^2\vec{k}$$

So

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t^3\vec{i} + 3\vec{j} + t^2\vec{k}) \cdot (\vec{i} + 2t\vec{j})\,dt$$

$$= \int_0^1 t^3 + 6t\,dt$$

$$= \left[\frac{t^4}{4} + 3t^2\right]_0^1$$

$$= \frac{1}{4} + 3 = 3\frac{1}{4}$$

(4) [10 pts] Consider the two vector fields

$$\begin{aligned} \mathbf{F}_1(x,y) &= (2xy - 2y^2\sin x)\mathbf{i} + (x^2 + 4y\cos x)\mathbf{j} \\ \mathbf{F}_2(x,y) &= (2xy^2 - 2y\sin x)\mathbf{i} + (x^2 + 4y^2\cos x)\mathbf{j} \end{aligned}$$

$= P_1\vec{i} + Q_1\vec{j}$

$= P_2\vec{i} + Q_2\vec{j}$

One of these vector fields is conservative.

(a) Which vector field is conservative and which is not? Why?

$\dfrac{\partial Q_1}{\partial x} = 2x - 4y\sin x \qquad \dfrac{\partial P_1}{\partial y} = 2x - 4y\sin x$

So $\dfrac{\partial Q_1}{\partial x} = \dfrac{\partial P_1}{\partial y}$ and $\vec{F}_1$ is defined on all of $\mathbb{R}^2$ which is simply connected. Therefore $\vec{F}_1$ is conservative

BUT $\dfrac{\partial Q_2}{\partial x} = 2x - 4y^2\sin x \qquad \dfrac{\partial P_2}{\partial y} = 4xy - 2\sin x$

So $\partial P_2/\partial y \neq \partial Q_2/\partial x$ So $\vec{F}_2$ NOT conservative

(b) For the vector field that is conservative, evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is any curve from $(0,0)$ to $(0,1)$.

Find f so that $\vec{F} = \nabla f$.

$\dfrac{\partial f}{\partial x} = P = 2xy - 2y^2\sin x \Rightarrow f(x,y) = x^2y + 2y^2\cos x + g_1(y)$

$\dfrac{\partial f}{\partial y} = Q = x^2 + 4y\cos x \Rightarrow f(x,y) = x^2y + 2y^2\cos x + g_2(x)$

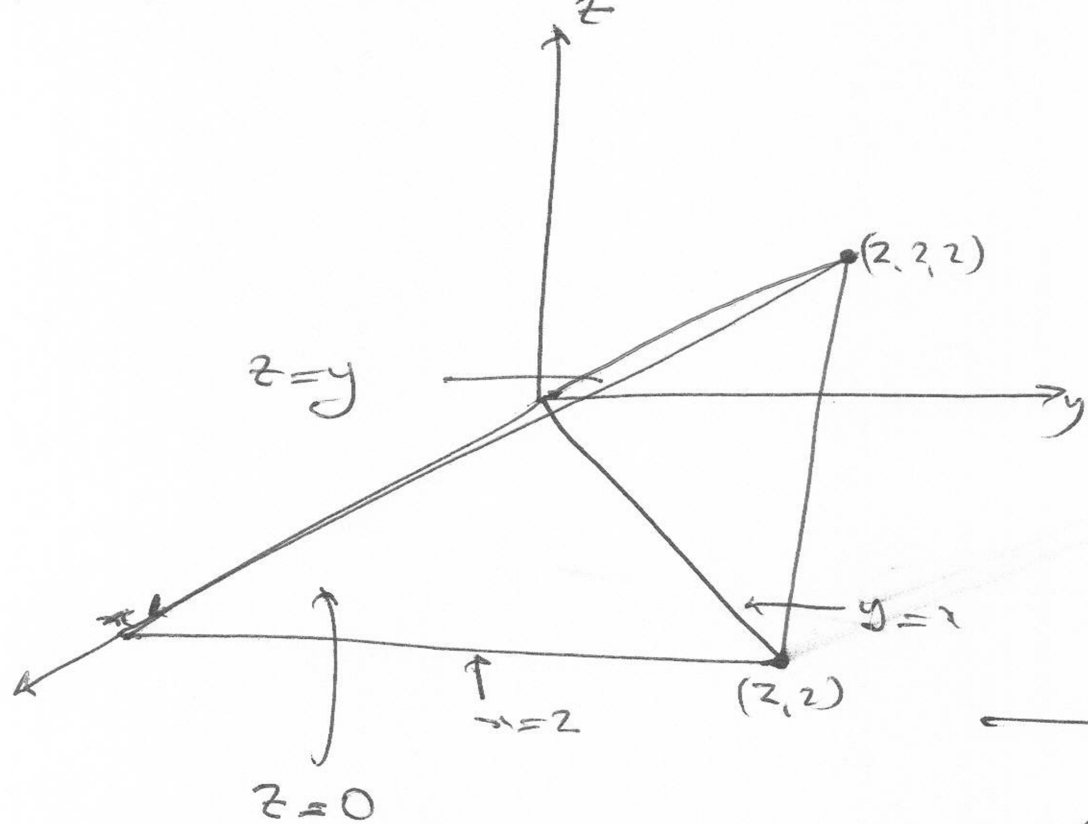
So $f(x,y) = x^2y + 2y^2\cos x$

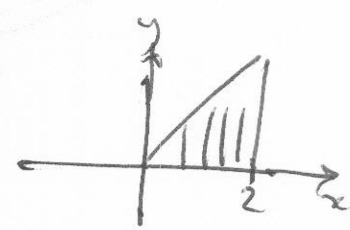
So by FTC for line integrals

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(0,1) - f(0,0)$

$= 2 - 0 = \underline{\underline{2}}$

(5) [10 pts] Find a double integral equal to the volume of the solid bounded by the surfaces $y = x$, $x = 2$, $z = 0$, and $z = y$, and evaluate this integral.





$0 \le x \le 2$

$0 \le y \le x$

$$\text{VOL} = \int_{x=0}^{x=2} \int_{y=0}^{y=x} (y - 0)\,dy\,dx$$

$$= \int_{x=0}^{x=2} \left[\frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^2 \frac{x^2}{2}\,dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________