NAME: SOLUTIONS

1 /10	2 /10	3 /8	4 /14	5 /8	T /50

MATH 251 (Fall 2010) Exam I, Sept 23rd

No calculators, books or notes! Show all work and give **complete explanations**. This 65 min exam is worth 50 points.

(1) [10 pts] Let $\mathbf{u} = (6, 0, 8)$ and $\mathbf{v} = (1, -2, 3)$ be two vectors in space.
(a) Calculate the vector projection, $\text{Proj}_{\mathbf{v}}(\mathbf{u})$, of the vector $\mathbf{u}$ onto the vector $\mathbf{v}$.

$$\text{PROJ}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$$

$$= \frac{(6,0,8)\cdot(1,-2,3)}{|(1,-2,3)|^2}(1,-2,3)$$

$$= \frac{6+24}{1^2+2^2+3^2}(1,-2,3) = \frac{30}{14}(1,-2,3) = \frac{15}{7}(1,-2,3)$$

(b) Find a vector that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

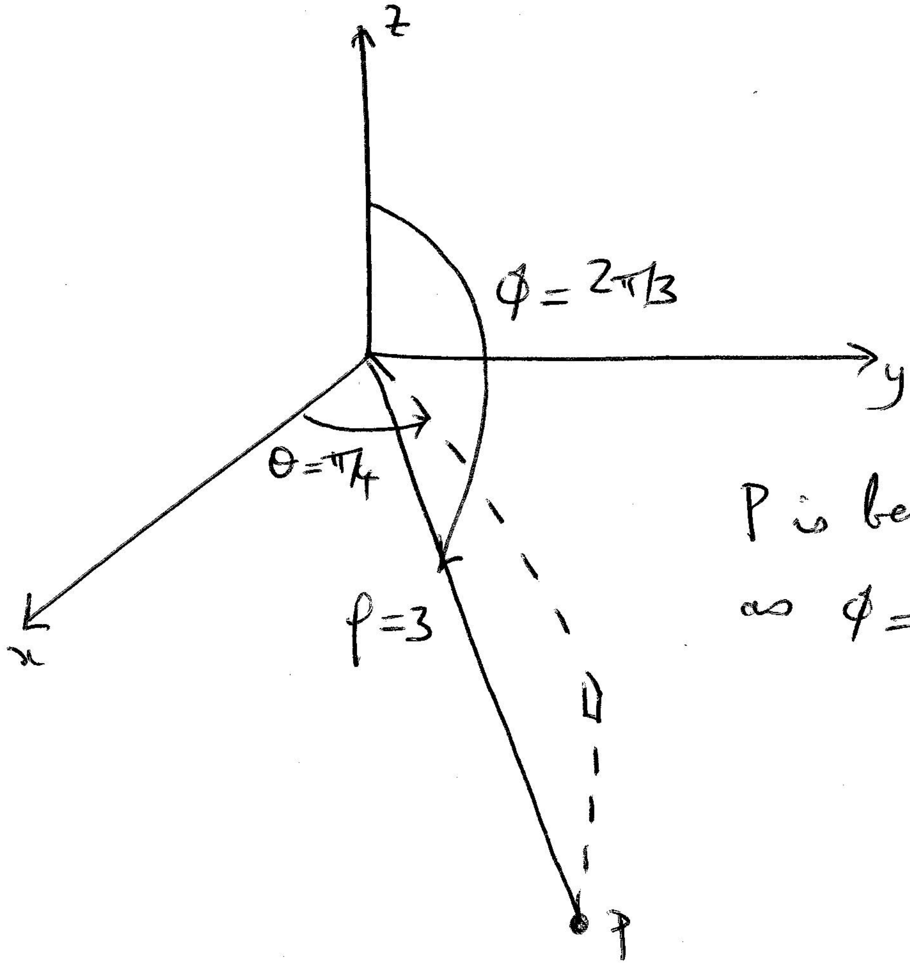
$\vec{w} = \vec{u}\times\vec{v}$ is $\perp$ to both $\vec{u}$ and $\vec{v}$.

$$\vec{w} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 6 & 0 & 8 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= (0\times3 + 2\times8)\vec{\imath} - (6\times3 - 1\times8)\vec{\jmath} + (6\times-2 + 1\times0)\vec{k}$$

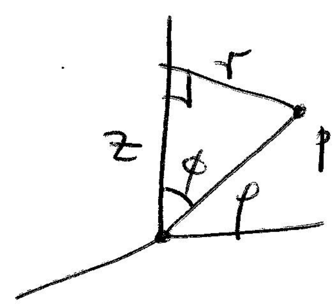
$$= 16\,\vec{\imath} - 10\,\vec{\jmath} - 12\,\vec{k}$$

(3) [8 pts] Let P be the point in space with spherical coordinates $(\rho, \theta, \phi) = (3, \frac{\pi}{4}, \frac{2\pi}{3})$. Sketch P and convert P to both rectangular and cylindrical coordinates.



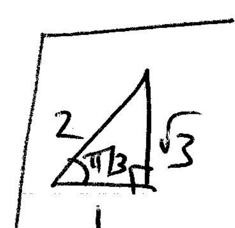
P is below xy-plane as $\phi = \frac{2\pi}{3} > \pi/2$.

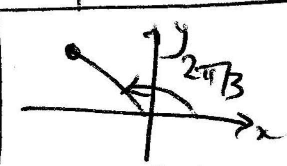
GENERAL PICTURE



$z = \rho \cos\phi$

$r = \rho \sin\phi$





$\cos 2\pi/3 < 0$

$\sin 2\pi/3 > 0$

CYL

$r = \rho \sin\phi = 3 \sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{2}$

$\theta = \frac{\pi}{4}$

$z = \rho \cos\phi = 3 \cos\frac{2\pi}{3} = -\frac{3}{2}$

RECT $x = r\cos\theta = \frac{3\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{3\sqrt{6}}{4}$

$y = r\sin\theta = \frac{3\sqrt{6}}{4}$

$z = -3/2$

(4) [14 pts] Find the traces (i.e., slices) of the surface

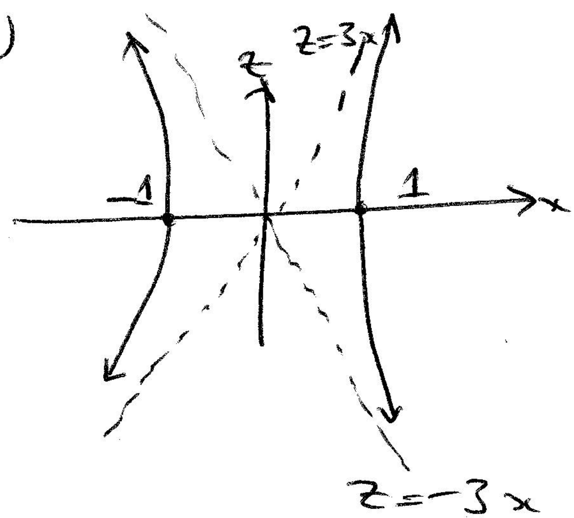
$$x^2 = 1 + \frac{y^2}{4} + \frac{z^2}{9}$$

in the planes $y = 0$, $z = 0$, and $x = k$, for $k = 0, \pm 1, \pm 2, \pm 3$. Then sketch the surface and name it.

$\boxed{y=0}$ $x^2 - \left(\frac{z}{3}\right)^2 = 1$ Hyperbola

Asymptotes are $x^2 - \left(\frac{z}{3}\right)^2 = 0$, ie $z = \pm 3x$.

Goes thru $(\pm 1, 0)$

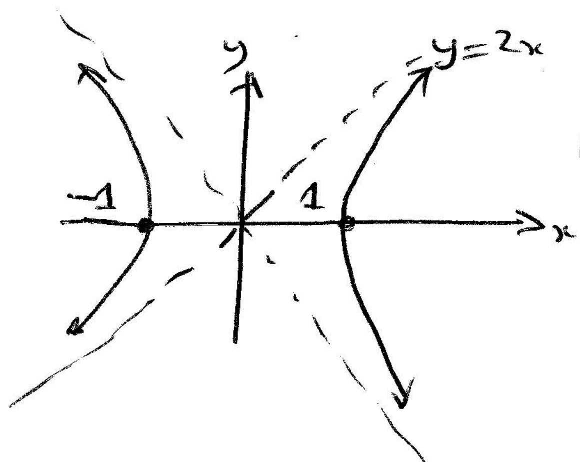


Steeper

$\boxed{z=0}$ $x^2 - \left(\frac{y}{2}\right)^2 = 1$

Asymptotes $y = \pm 2x$.

Goes thru $(\pm 1, 0)$



Less Steep

$\boxed{x = \pm k}$ $\left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = k^2 - 1$

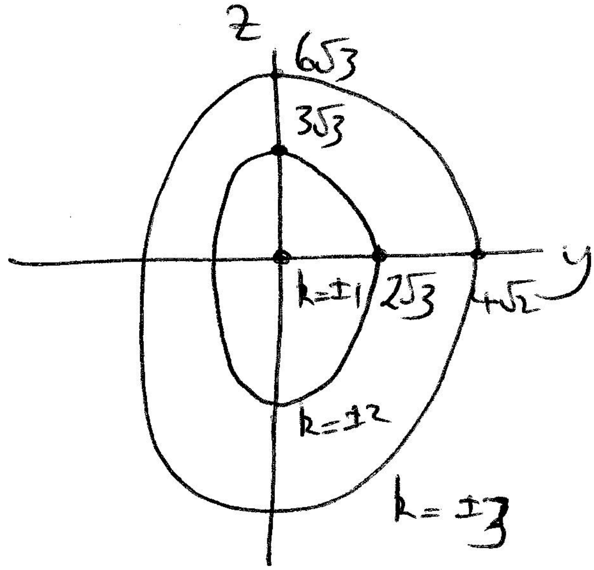
$\underline{k=0}$ NO SOLUTIONS

$\underline{k=\pm 1}$ $\left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = 0$, So $(y,z) = (0,0)$ ORIGIN.

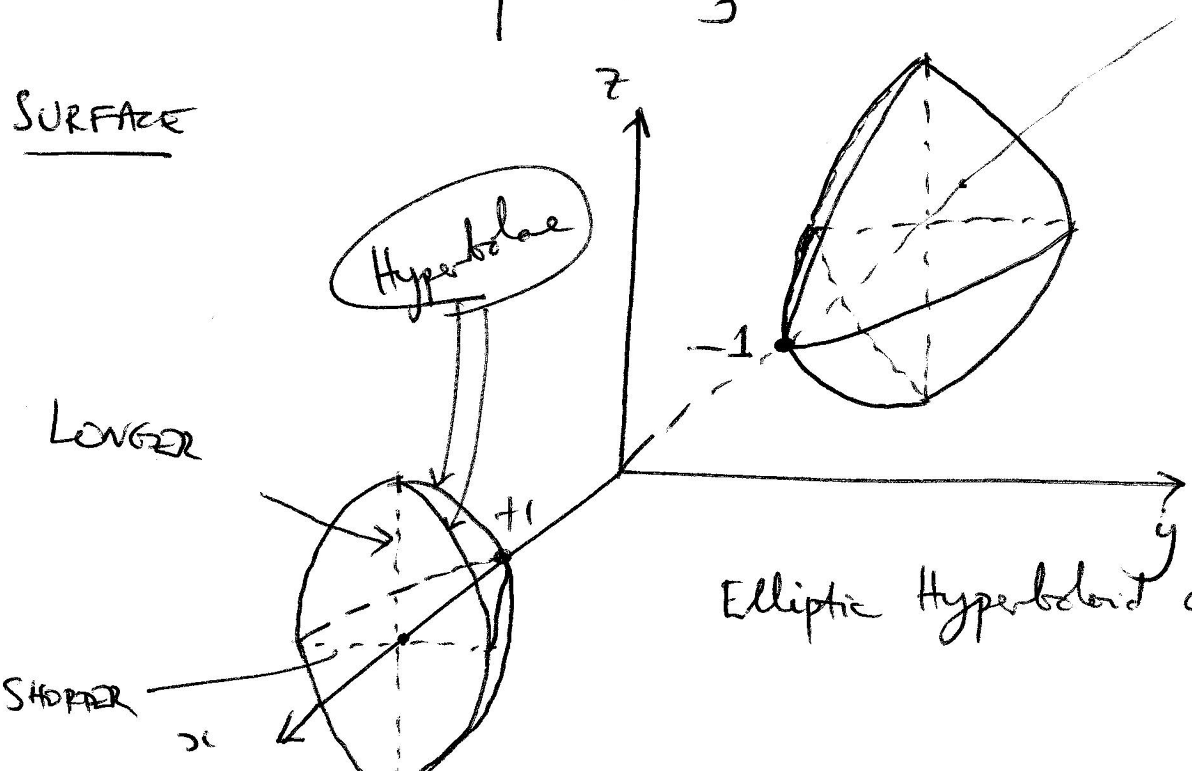
$\underline{k = \pm 2}$ $\left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = 3$

So $\left(\frac{y}{2\sqrt{3}}\right)^2 + \left(\frac{z}{3\sqrt{3}}\right)^2 = 1$ Ellipse

$\underline{k = \pm 3}$ $\left(\frac{y}{4\sqrt{2}}\right)^2 + \left(\frac{z}{6\sqrt{2}}\right)^2 = 1$ Ellipse



<u>SURFACE</u>



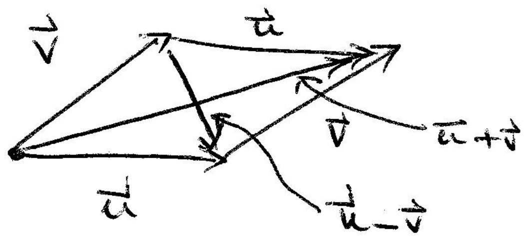
Elliptic Hyperboloid of 2 Sheets

(5) [8 pts] The Parallelogram Law states that, for any vectors **u** and **v**,

$$|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2.$$

(a) Give a geometrical interpretation of the Parallelogram Law.

The diagonals of the parallelogram determined by $\vec{u}, \vec{v}$ are $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.



<u>$\parallel$GRAM LAW SAYS</u>

The sum of squares of lengths of diagonals of any $\parallel$ gram equals sum of squares of lengths of 4 sides.

(b) Prove the Parallelogram Law using vector algebra. [Hint: Use $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$ together with the distributive law for the dot product.]

$$|\vec{u}+\vec{v}|^2 + |\vec{u}-\vec{v}|^2$$

$$= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$+ \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= 2|\vec{u}|^2 + 2|\vec{v}|^2 \quad \text{as } |\vec{u}|^2 = \vec{u}\cdot\vec{u}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________

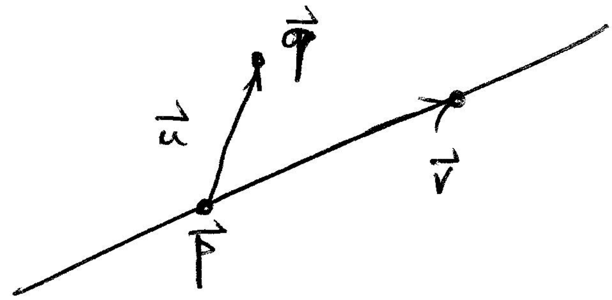
(2) [10 pts]
(a) Find a parametrization of the plane that contains both the point $(2, 4, 6)$ and the line $x = 7 - 3t$, $y = 3 + 4t$, $z = 5 + 2t$.

$\vec{l}(t) = (7-3t, 3+4t, 5+2t) = \vec{p} + t\vec{v}$

$\vec{q} = (2, 4, 6)$

$\vec{p} = \vec{l}(0) = (7, 3, 5)$

$\vec{v} = \vec{l}\,'(0) = (-3, 4, 2)$



e plane goes thru endpoint of $\vec{p}$ and contains

e vectors $\vec{u} = \cancel{\vec{p}-\vec{q}}\ \vec{q}-\vec{p}$ and $\vec{v}$. So a parametrization

f this plane is $\qquad \vec{u} = (-5, 1, 1)$

$$\vec{r}(s,t) = \vec{p} + s\vec{u} + t\vec{v} = \vec{l}(t) + s\vec{u}$$

$$= (7-3t-5s,\ 3+4t+s,\ 5+2t+s) \qquad s, t \in \mathbb{R}$$

(b) Find a level set equation (i.e., an equation of the form $ax + by + cz = d$) for the plane in (a).

normal vector to the plane is $\vec{n} = \vec{u} \times \vec{v}$

he plane contains the point $\vec{p}$.

o Level Set Equation is $(\vec{r} - \vec{p}) \cdot \vec{n} = 0$ where $\vec{r} = (x, y, z)$

ince $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 1 & 1 \\ -3 & 4 & 2 \end{vmatrix} = (-2, 7, -17)$

e level set equation is

$$-2(x-7) + 7(y-3) - 17(z-5) = 0$$