NAME: SOLUTIONS

1 /6	2 /12	3 /10	4 /12	5 /10	T /50

MATH 251 (Fall 2010) Exam II, Oct 21st

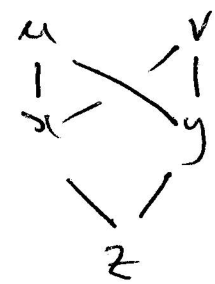
No calculators, books or notes! Show all work and give **complete explanations**. This 65 min exam is worth 50 points.

(1) [6 pts] Suppose that

$$x = 3u + 2v, \qquad y = 4u - 5v$$

and let $z = f(x, y)$ be a function so that

(a,b)	$f(a,b)$	$\frac{\partial f}{\partial x}(a,b)$	$\frac{\partial f}{\partial y}(a,b)$
$(1,2)$	5	6	4
$(7,-6)$	-1	-5	7



Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at $(u, v) = (1, 2)$.

$$z = f(x(u,v),\ y(u,v))$$

$$\frac{\partial z}{\partial u}(1,2) = \frac{\partial z}{\partial x}(x(1,2), y(1,2))\,\frac{\partial x}{\partial u}(1,2) + \frac{\partial z}{\partial y}(x(1,2), y(1,3))\,\frac{\partial y}{\partial u}(1,2)$$

$$= \frac{\partial z}{\partial x}(7,-6)\cdot 3 + \frac{\partial z}{\partial y}(7,-6)\, 4$$

$$= -5 \times 3 + 7 \times 4 = 13$$

$$\frac{\partial z}{\partial v}(1,2) = \frac{\partial z}{\partial x}(7,-6)\,\frac{\partial x}{\partial v}(1,2) + \frac{\partial z}{\partial y}(7,-6)\,\frac{\partial y}{\partial v}(1,2)$$

$$= 2 \times (-5) + 7 \times (-5) = -45$$

(2) [12 pts]

(a) Sketch the parametrized curve $(x, y) = r(t) = (2\sin t, 3\cos t)$ for $0 \le t \le \pi$.

$x = 2\sin t, \quad y = 3\cos t$

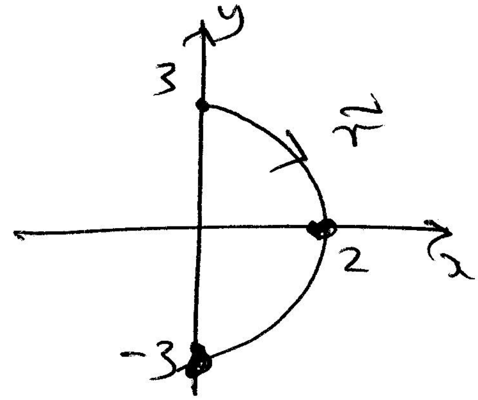
So $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$

Curve is (part of) an ellipse

$\vec{r}(0) = (0,3)$

$\vec{r}(\pi/2) = (2,0)$

$\vec{r}(\pi) = (0,-3)$



RIGHT HALF OF ELLIPSE

(b) Sketch the level curves of the function $z = f(x, y) = x - e^y$ at levels $k = -1$, $k = 0$, and $k = 1$. Also calculate the gradient of f at the origin, add it to your sketch, and explain how it is related to the level curve that passes through the origin.

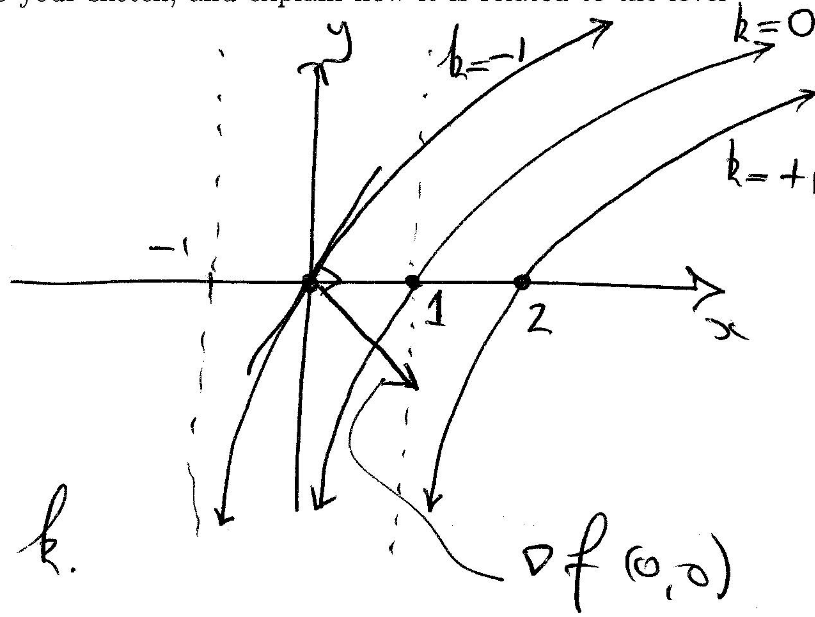
$z = f(x,y) = x - e^y$

$\boxed{z = k}$ $\quad x - e^y = k$

$e^y = x - k$

$y = \ln(x-k)$

Shift $y = \ln x$ to right by k.



$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = 1\vec{i} - e^y\vec{j}$

$\nabla f(0,0) = 1\vec{i} - 1\vec{j} = \vec{i} - \vec{j} = (1,-1)$

$\nabla f(0,0)$ is perpendicular to the tangent line to the level curve of $z = f(x,y)$ through $(0,0)$.

(3) [10 pts] Let $z = f(x, y) = x^2 + y^3 + 4xy$.

(a) Suppose that the function $z = f(x, y)$ is temperature at the point (x, y) in the plane. Suppose that a stink bug is walking at constant speed in this plane. In what direction should the stink bug walk from the point $(x, y) = (-1, 2)$ to decrease its temperature the fastest?

$\nabla f(x,y) = (2x + 4y,\ 3y^2 + 4x)$

$\nabla f(-1,2) = (-2+8,\ 12-4) = (6, 8)$

Direction is $\vec{u} = \dfrac{-\nabla f(-1,2)}{|\nabla f(-1,2)|} = \dfrac{(-6,-8)}{\sqrt{36+64}} = \left(\dfrac{-3}{5}, \dfrac{-4}{5}\right)$

(b) Find the rate of change of f at the point $(x, y) = (-1, 2)$ in the direction of the vector $2\mathbf{i} + 3\mathbf{j}$.

Let $\vec{u} = \dfrac{2\vec{i} + 3\vec{j}}{|2\vec{i} + 3\vec{j}|} = \dfrac{(2,3)}{\sqrt{13}}$

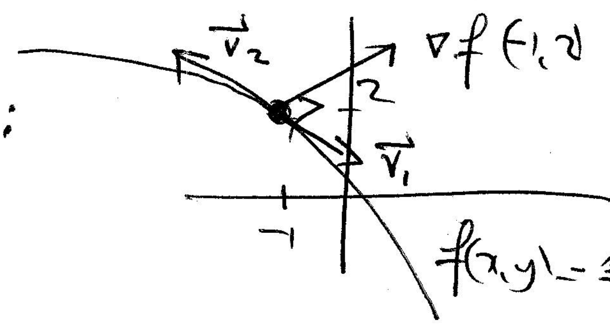
Rate of Change of f in direction $\vec{u}$ at $(-1,2)$

$= D_{\vec{u}} f(-1,2) = \nabla f(-1,2) \cdot \vec{u}$

$= (6,8) \cdot \dfrac{(2,3)}{\sqrt{13}} = \dfrac{36}{\sqrt{13}}$

(c) Find a vector that is *tangent* to the level curve $x^2 + y^3 + 4xy = 1$ at the point $(x, y) = (-1, 2)$.

To Find $\vec{v}_1$ or $\vec{v}_2$ so that $\vec{v}$ is tangent as in picture;



Use fact $\nabla f(-1,2)$ is $\perp$ to level curve.

So $\vec{v} \cdot \nabla f(-1,2) = 0$ must hold

$\vec{v} \cdot (6, 8) = 0.$

$\vec{v} = (-8, 6)$ does the job:

[illegible]

(4) [12 pts] Let S be the surface parametrized by

$$\mathbf{r}(u,v) = (1+\cos u, \sin u, v) \qquad 0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 4.$$

(a) Find a level-set equation of the form $F(x,y,z) = 0$ that is satisfied by all points on S.

$x = 1 + \cos u$

$y = \sin u$

$z = v$

$$(x-1)^2 + y^2 = \cos^2 u + \sin^2 u = 1$$

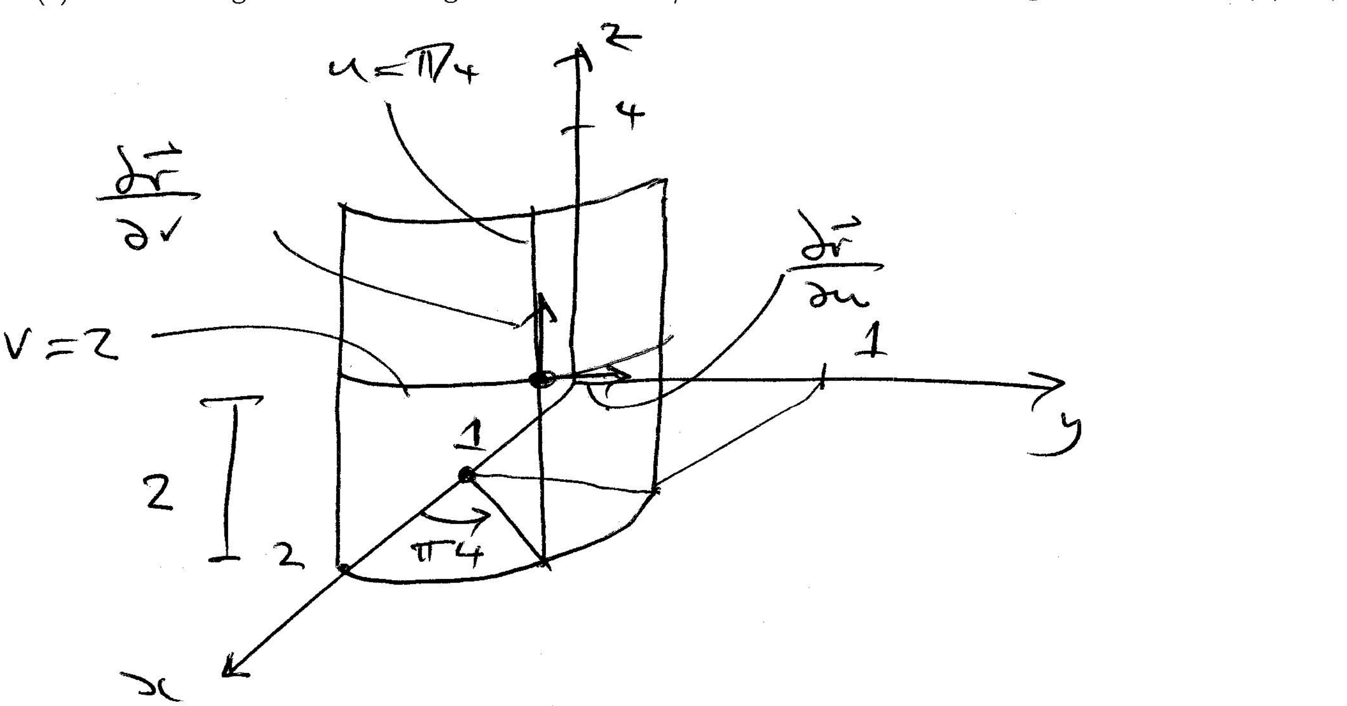
$$F(x,y,z) = (x-1)^2 + y^2 - 1.$$

(b) Calculate the tangent vectors to the grid curves $u = \pi/4$ and $v = 2$ at the point $\mathbf{r}(\pi/4, 2)$.

$$\frac{\partial \vec{r}}{\partial u} = (-\sin u, \cos u, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \text{ at } \vec{r}(\pi/4, 2)$$

$$\frac{\partial \vec{r}}{\partial v} = (0, 0, 1).$$

(c) Sketch S together with the grid curves $u = \pi/4$ and $v = 2$ and their tangent vectors at $\mathbf{r}(\pi/4, 2)$.



(5) [10 pts] Find all local maxima, local minima, and saddle points of the function $z = f(x, y) = xye^y$.

$$z = f(x,y) = xy\,e^y$$

$$\nabla f = (ye^y, \; x(e^y + ye^y))$$

$$= (y\,e^y, \; x(1+y)\,e^y) = (0,0)$$

at $y = 0$ and $x = 0$. $\boxed{(0,0)}$

$$D = \det \begin{bmatrix} 0 & (1+y)\,e^y \\ (1+y)e^y & x(2+y)\,e^y \end{bmatrix} = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

at $(0,0)$

So Saddle Point at $(0,0)$

as $D = -1 < 0$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________