NAME: SOLUTIONS

1 /11	2 /12	3 /10	4 /12	5 /5	T /50

MATH 251 (Fall 2010) Exam III, Nov 23rd

No calculators, books or notes! Show all work and give **complete explanations**. This 70 min exam is worth 50 points.

(1) [11 pts]
Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}$ is the vector field $\mathbf{F}(x,y,z) = (x+yz)\mathbf{i}+2x\mathbf{j}+xyz\mathbf{k}$ and C is the line segment from $(1,0,1)$ to $(2,3,1)$.

$$\vec{r}(t) = (1,0,1) + t[(2,3,1)-(1,0,1)] = (1,0,1)+t(1,3,0) \quad 0\le t\le 1$$
$$\vec{r}\,'(t) = (1,3,0) \qquad\qquad = (1+t, 3t, 1)$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$
$$= \int_0^1 (1+t+3t,\ 2(1+t),\ (1+t)(3t)1)\cdot(1,3,0)\,dt$$
$$= \int_0^1 1(1+4t) + 6(1+t)\,dt = \int_0^1 10t+7\,dt$$
$$= [5t^2+7t]_0^1 = 12$$

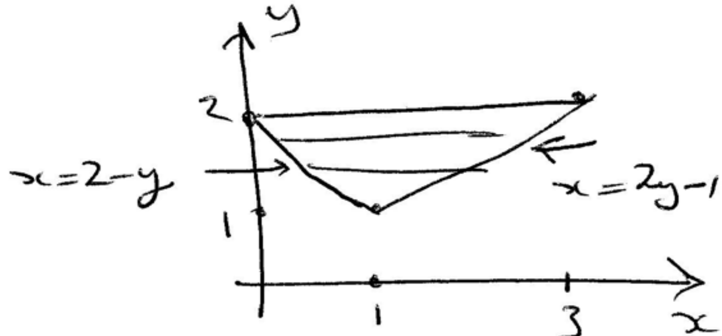
(b) Carefully state Green's Theorem (a picture might help!).

Let D be a domain in the plane with boundary curve, ∂D. Orient ∂D so that as go around in direction of orientation with head in $+z$ direction, D is on your left.
Let $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a VF on D. Then
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy.$$

(2) [12 pts]

(a) Calculate $\iint_D y^3\,dA$, where D is the triangle with vertices $(0,2)$, $(1,1)$, and $(3,2)$.

D HAS HORIZONTAL TOP + BOTTOM.



So $1 \le y \le 2$

$2-y \le x \le 2y-1$

$$\iint_D y^3\,dA = \int_{y=1}^{y=2}\int_{x=2-y}^{x=2y-1} y^3\,dx\,dy$$

$$= \int_{y=1}^{y=2} \left[x y^3\right]_{x=2-y}^{x=2y-1}\,dy$$

$$= \int_1^2 \left[(2y-1)-(2-y)\right]y^3\,dy = 3\int_1^2 (y-1)y^3\,dy$$

$$= 3\int_1^2 y^4 - y^3\,dy = 3\left[\frac{y^5}{5}-\frac{y^4}{4}\right]_1^2$$

$$= 3\left[2^4\left(\frac{2}{5}-\frac{1}{4}\right)-\left(\frac{1}{5}-\frac{1}{4}\right)\right] = 3\left[16.\frac{8-5}{20} - \frac{4-5}{20}\right]$$

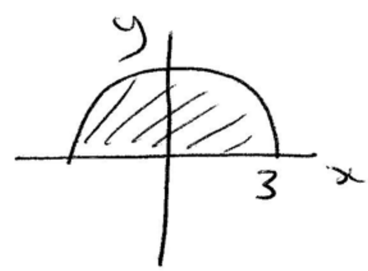
$$= \frac{3\times 49}{20} = 147/20$$

(b) Calculate $\iint_D \cos(x^2+y^2)\,dA$, where D is the region above the x-axis and within the circle $x^2+y^2=9$.

In polar coords D is

$0 \le \theta \le \pi$

$0 \le r \le 3$



So

$$\iint_D \cos(x^2+y^2)\,dA = \int_{\theta=0}^{\pi}\int_{r=0}^{3} \cos(r^2)\,r\,dr\,d\theta$$

$$= \pi\int_{r=0}^{3}\cos(r^2)\,r\,dr = \frac{\pi}{2}\int_{u=0}^{9}\cos(u)\,du = \frac{\pi}{2}\left[\sin(u)\right]_0^9$$

$u = r^2$

$$= \frac{\pi}{2}\sin(9)$$

(3) [10 pts] Consider the two vector fields

$$\begin{aligned}\mathbf{F}_1(x,y) &= (3x-2y)\mathbf{i} \;+\; (-4x+3y-8)\mathbf{j}\\ \mathbf{F}_2(x,y) &= (2x-3y)\mathbf{i} \;+\; (-3x+4y-8)\mathbf{j}\end{aligned}$$

One of these vector fields is conservative.

(a) Which vector field is conservative and which is not? Why?

$\boxed{\vec{F}_1}$ $\quad \frac{\partial Q}{\partial x} = -4, \quad \frac{\partial P}{\partial y} = -2. \quad \frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$ NOT CONS.

$\boxed{\vec{F}_2}$ $\quad \frac{\partial Q}{\partial x} = -3 \quad \frac{\partial P}{\partial y} = -3. \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ AND

$\vec{F}_2$ is defined on all of $\mathbb{R}^2$ which is simply connected. So $\vec{F}_2$ is conservative

(b) For the vector field that is conservative, evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any curve from $(0,0)$ to $(2,0)$.

$$I = \int_C \vec{F}_2 \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,0) - f(0,0)$$

To find f:

$$\frac{\partial f}{\partial x} = 2x - 3y \quad \Rightarrow \quad f(x,y) = x^2 - 3yx + g(y)$$

$$\frac{\partial f}{\partial y} = -3x + 4y - 8 \quad \Rightarrow \quad f(x,y) = -3xy + 2y^2 - 8y + h(x)$$

So $\quad f(x,y) = x^2 - 3xy - 8y + 2y^2 + c.$

So

$I = 4 - 0 = \boxed{4}$

(4) [12 pts]
(a) Use the method of Lagrange Multipliers to find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2$ on the ellipse $(x-1)^2 + 4y^2 = 4$.

$$\left\{\begin{array}{l}\nabla f = \lambda \nabla g \\ g = c\end{array}\right\} \quad \text{gives} \quad \begin{array}{ll} 2x = \lambda\, 2(x-1) & (1) \\ 2y = \lambda\, 8y & (2) \\ (x-1)^2 + 4y^2 = 4 & (3) \end{array}$$

By (2) $\quad y(1-4\lambda) = 0$

So $\quad y=0$ or $\lambda = \frac{1}{4}$

$\boxed{y=0}$ By (3) $(x-1)^2 = 4, \quad x-1 = \pm 2, \quad x = 3, -1$

So $\boxed{(x,y) = (3,0)}$ $\quad \boxed{(x,y) = (-1,0)}$

$\lambda = \frac{x}{x-1} = \frac{3}{2} \qquad \lambda = \frac{-1}{-2} = \frac{1}{2}.$

OR (A) $\boxed{f(3,0) = 9}$ ABS MAX (B) $\boxed{f(-1,0) = 1}$ ~~ABS MIN~~

OR $\boxed{\lambda = \frac{1}{4}}$

By (1) $\quad 4x = x - 1$

$3x = -1$

$x = -1/3.$

So by (3) $\quad \frac{16}{9} + 4y^2 = 4$

$1 - \frac{4}{9} = y^2 \qquad y = \pm \frac{\sqrt{5}}{3}$

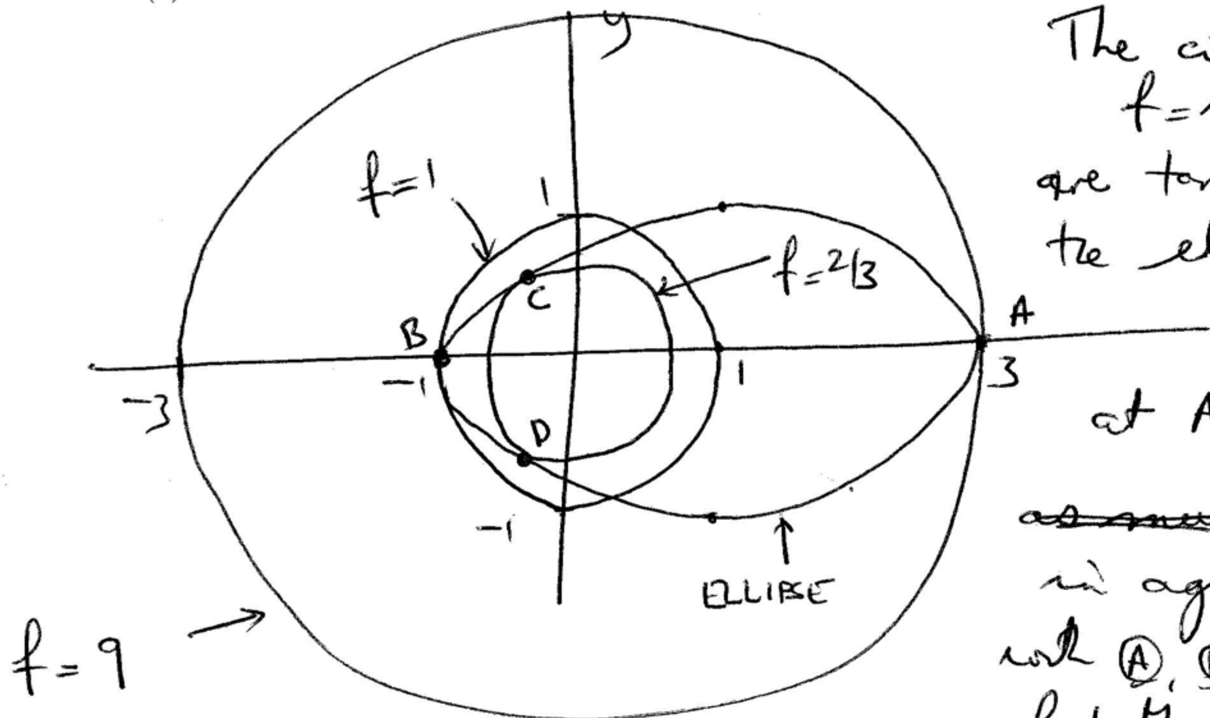
So $(x,y) = \left(-\frac{1}{3}, \pm\frac{\sqrt{5}}{3}\right), \lambda = \frac{1}{4}$

$f\left(-\frac{1}{3}, \pm\frac{\sqrt{5}}{3}\right) = \frac{1}{9} + \frac{5}{9} = \frac{2}{3}$

(C) + (D)

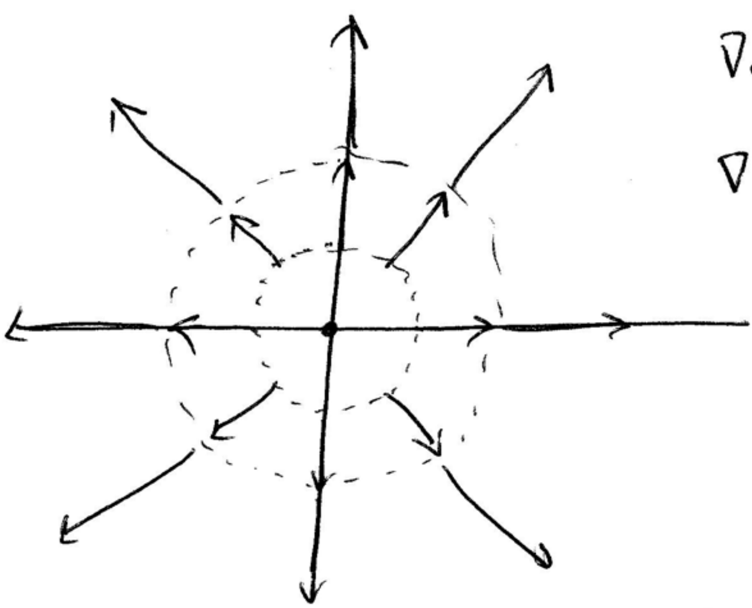
ABS MIN

(b) By sketching the ellipse and some appropriately chosen level curves, $f(x, y) = k$, determine the approximate locations of the absolute maxima and minima of f on the ellipse, and compare to the answer you found in (a).



The circles $f=k$ are tangent to the ellipse at A, B, C, D ~~[illegible]~~ in agreement with (A), (B), (C), (D) of L.M. calculation

(5) [5 pts] Sketch a vector field $\mathbf{F}$ in the plane so that $\mathbf{F}(0,0) = (0,0)$, $(\nabla \times \mathbf{F})(0,0) = (0,0)$, and $(\nabla \cdot \mathbf{F})(0,0) > 0$.



$\vec{F}(x,y) = x\vec{i} + y\vec{j}$ POSITION VF

$\nabla \cdot \vec{F} = 1 + 1 = 2 > 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \vec{0}$$

IF $\vec{F}$ is Velocity VF of fluid FLUID $\underline{\text{EXPANDS}}$ (DIVERGES) away from $\vec{0}$

So $\text{DIV}(\vec{F})(0,0) > 0$.

FLUID does NOT rotate about $\vec{x}$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________