NAME: SOLUTIONS

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MATH 251 (Fall 2010) Final Exam, Dec 16th

No calculators, books or notes! Show all work and give **complete explanations**. This 120 min exam is worth 100 points.

(1) [8 pts] Calculate the following limits or show they do not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-3y^2}{4x^2+7y^2}$

ALONG $x=0$: $\lim_{y\to 0} \frac{-3y^2}{7y^2} = \lim_{y\to 0} -\frac{3}{7} = -\frac{3}{7}$

ALONG $y=0$ $\lim_{x\to 0} \frac{x^2}{4x^2} = \lim_{x\to 0} \frac{1}{4} = \frac{1}{4}$

Since $-3/7 \neq 1/4$ limit DNE

(b) $\lim_{(x,y)\to(0,0)} \frac{2+x+y}{1+x^2+y^2} = \frac{2+0+0}{1+0^2+0^2} = 2$

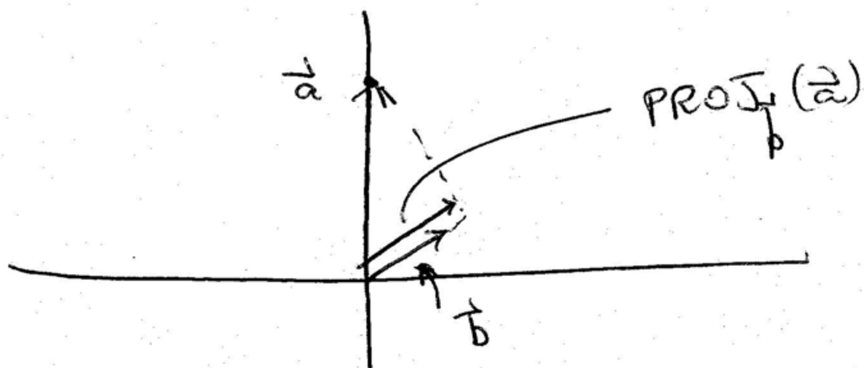
Since the denominator is not zero at $(0,0)$ we can just plug $(x,y)=(0,0)$ in,

ie we can use fact the function is defined + continuous at $(x,y)=(0,0)$

(2) [12 pts]
(a) Calculate the projection of the vector $\mathbf{a} = 3\mathbf{j}$ onto the vector $\mathbf{b} = \mathbf{i} + \mathbf{j}$. Draw a labelled picture that clearly illustrates the relationship between these three vectors.

$$\mathrm{PROJ}_{\vec{b}}(\vec{a}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\frac{\vec{b}}{|\vec{b}|} = \frac{3\vec{j}\cdot(\vec{i}+\vec{j})}{\sqrt{2}}\ \frac{\vec{i}+\vec{j}}{\sqrt{2}}$$

$$= \frac{3}{2}(\vec{i}+\vec{j})$$



(b) Find the volume of the parallelipiped determined by the vectors $(1, 2, 3)$, $(0, 1, -4)$, and $(5, 0, 2)$.

$$\mathrm{VOL} = \left| \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 5 & 0 & 2 \end{vmatrix} \right|$$

$$= \left| 1\,(1\times 2 - 0\times(-4)) - 2\,(0\times 2 + 5\times 4) + 3\,(0\times 0 - 5\times 1) \right|$$

$$= |2 - 40 - 15| = |-53| = 53$$

(3) [10 pts]
(a) Find an equation of the form $z = ax+by+c$ for the tangent plane to the graph of $z = f(x,y) = 3x^2+5y^2$ at the point $(x,y,z) = (1,2,23)$.

$$\frac{\partial f}{\partial x} = 6x = 6 \quad @ \quad (x,y) = (1,2)$$

$$\frac{\partial f}{\partial y} = 10y = 20 \quad @ \quad (x,y) = (1,2)$$

$$z = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$$

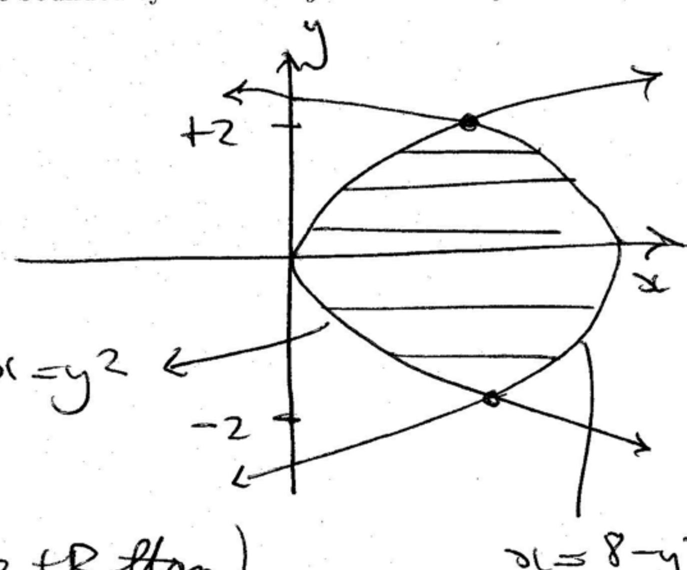
$$= 23 + 6(x-1) + 20(y-2)$$

$$z = 6x + 20y - 23$$

(b) Calculate $\iint_D y^2\,dA$, where D is the region in the xy-plane bounded by the curves $y^2 = x$ and $x+y^2 = 8$.

$x = y^2$ and $x = 8-y^2$ meet at

$$y^2 = 8-y^2$$
$$y^2 = 4$$
$$y = \pm 2$$



Region is Type II (Horizontal Top + Botton)

$$-2 \le y \le 2$$
$$y^2 \le x \le 8-y^2$$

So $$\iint_D y^2\,dA = \int_{y=-2}^{y=2} \int_{x=y^2}^{x=8-y^2} y^2\,dx\,dy = \int_{-2}^{2} y^2[(8-y^2)-y$$

$256/15$

(4) [6 pts] Calculate the length of the curve $\mathbf{r}(t) = (4t, 3\cos t, 3\sin t)$ for $0 \le t \le \pi$.

$$L = \int_0^{\pi} |\vec{r}\,'(t)|\,dt \qquad \vec{r}\,'(t) = (4, -3\sin t, 3\cos t)$$

$$= \int_0^{\pi} 5\,dt \qquad |\vec{r}\,'(t)| = \sqrt{4^2+3^2} = 5$$

$$= 5\pi$$

(5) [8 pts] An *anemometer* is an instrument that measures wind speed. Suppose that $\mathbf{F}(x,y) = y\mathbf{i} + 2x\mathbf{j}$ is the velocity vector field of air moving across the xy-plane. Suppose an ant that is carrying an anemometer is at the point $\mathbf{p} = (-1, 4)$ and is walking with velocity $\mathbf{v} = (2, 3)$. Is the wind speed measured by the anemometer increasing or decreasing?

Let $(x,y) = \vec{r}(t)$ be path of ant. So $\vec{r}(0) = \vec{p}$, $\vec{r}\,'(0) = \vec{v}$

The speed of wind is

$$V(x,y) = |\vec{F}(x,y)| = \sqrt{y^2+4x^2}$$

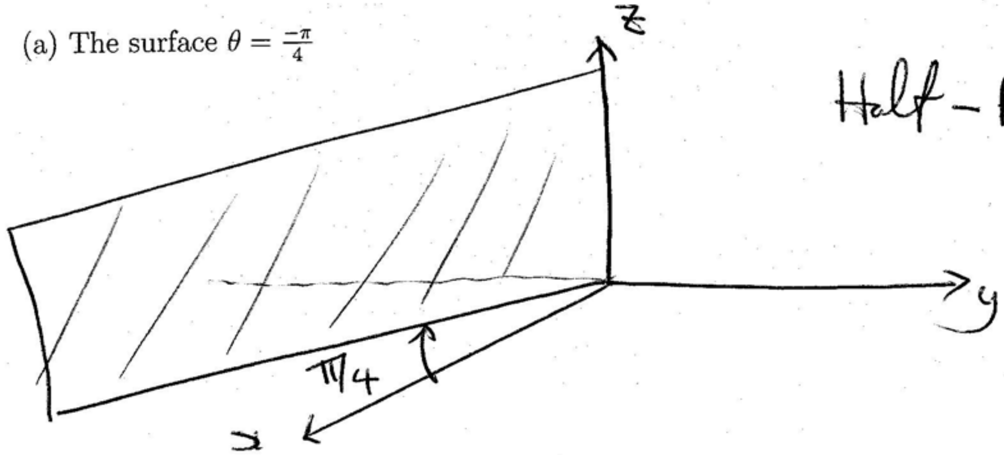
So $s(t) = V(\vec{r}(t))$ is speed of wind as measured by anemometer (NOTE I am ignoring the motion of ant relative to ground here. Strictly speaking it should be included — the question was poorly worded in this respect)

Then
$$s'(t) = \nabla V(\vec{r}(t)) \cdot \vec{r}\,'(t)$$
$$s'(0) = \nabla V(\vec{p}) \cdot \vec{v}$$

$$\nabla V = \tfrac{1}{2}(y^2+4x^2)^{-1/2}(8x, 2y)$$
$$\nabla V(\vec{p}) = \tfrac{1}{2}(20)^{-1/2}(-8, 8)$$
So
$$s'(0) = \frac{1}{2\sqrt{20}}(-8, 8)\cdot(2, 3)$$
$$= \frac{1}{\ldots}\, 8 > 0 \quad \text{INCREASIN[G]}$$

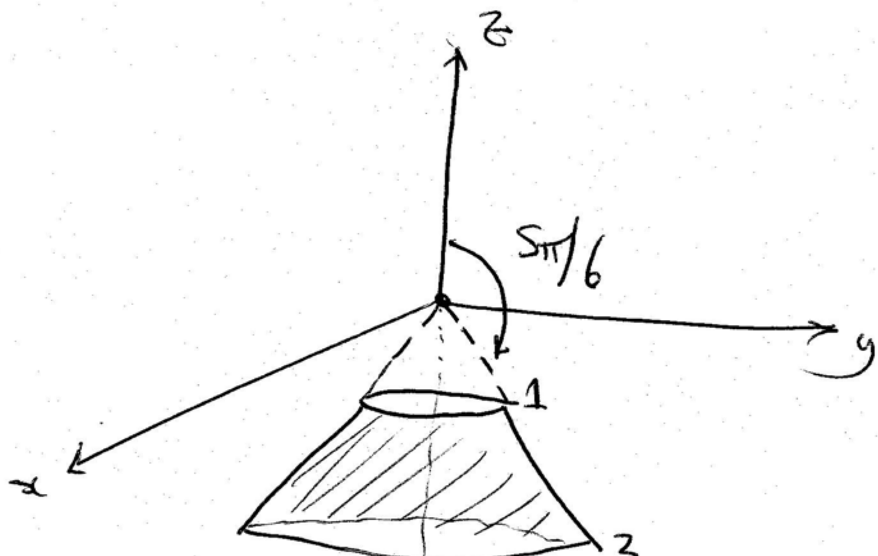
(6) [12 pts] Sketch the following.

(a) The surface $\theta = \frac{-\pi}{4}$



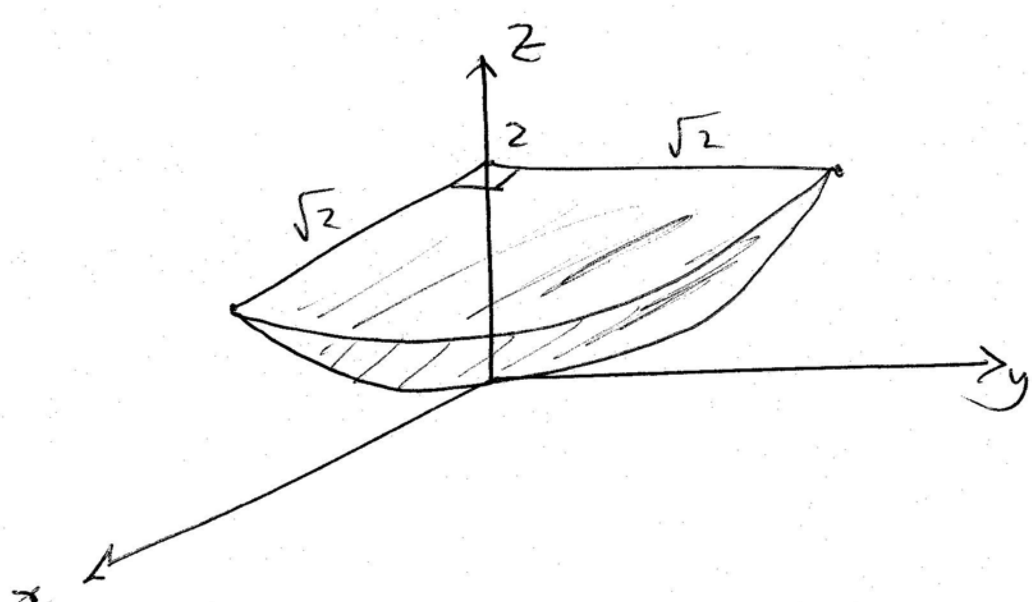
Half-Plane

(b) The surface $\phi = \frac{5\pi}{6}$ for $0 \leq \theta \leq 2\pi$ and $1 \leq \rho \leq 2$.

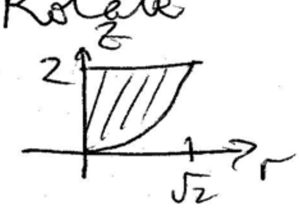


SURFACE IS A LAMPSHADE

(c) The solid region $0 \leq \theta \leq \frac{\pi}{2}$, $r^2 \leq z \leq 2$.



Rotate



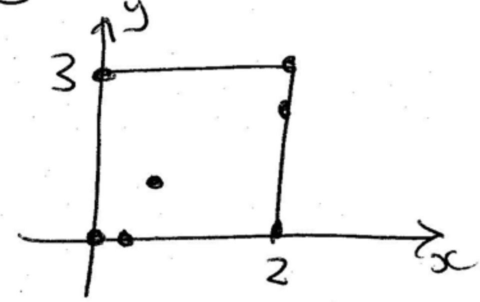
about z axis.

(7) [12 pts] Find the absolute maximum and minimum of the function $f(x, y) = 2x^2 + y^2 - 2xy - 2x$ on the rectangular region bounded by the lines $x = 0$, $y = 0$, $x = 2$, and $y = 3$.

Critical Points are

① Critical Points of f inside rectangle

② Critical Points on the 4 boundary curves and at their endpoints.



① $\nabla f(x,y) = (4x-2y-2,\ 2y-2x) = (0,0)$

when $y = x$ and $2x - y - 1 = 0$

So $2x - x - 1 = 0$, $x = 1$

$(1,1)$

② $\boxed{x=0}$ $g(y) = f(0, y^2) = y^2$; $0 \le y \le 3$

$(0,0)$, $(0,3)$

$\boxed{y=0}$ $h(x) = f(x,0) = 2x^2 - 2x$
$0 \le x \le 2$

$h'(x) = 4x - 2 = 0$ at $x = 1/2$

$(1/2, 0)$, $(0,0)$, $(2,0)$

$\boxed{x=2}$ $g(y) = f(2,y) = y^2 - 4y + 4$
$0 \le y \le 3$

$g'(y) = 2y - 4 = 0$ at $y = 2$

$\boxed{y=3}$ $h(x) = f(x,3) = 2x^2 - 8x + 9$, $0 \le x \le 2$.

$h'(x) = 4x - 8 = 0$ at $x = 2$

(x,y)	$f(x,y)$	
$(1,1)$	-1	MI
$(0,0)$	0	
$(0,3)$	9	MA
$(1/2, 0)$	$-1/2$	
$(2,0)$	4	
$(2,2)$	0	
$(2,3)$	1	

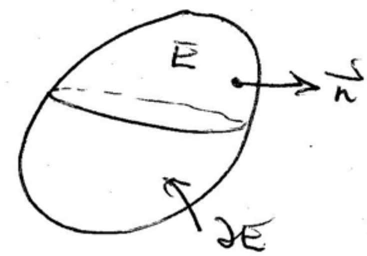
(8) [12 pts]

(a) Carefully state the Divergence Theorem. You may find it helpful to draw a picture and refer to it in your written explanation.

Let E be a solid region in space with boundary surface ∂E endowed with the outward normal vector. Let $\vec{F}$ be a vector field on $\bar{E}$.

Then

$$\iiint_E (\nabla \cdot \vec{F})\, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$



(b) Let S be the surface $x^2 + y^2 + z^2 = 4$ with the outward orientation, and let $\mathbf{F}$ be the vector field $\mathbf{F} = xz^2\mathbf{i} + \sin(z)\mathbf{j} + xy\mathbf{k}$. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$S = \partial E$ where E is ball of radius 2 center origin

$\nabla \cdot \vec{F} = z^2$.

So by DIV THM

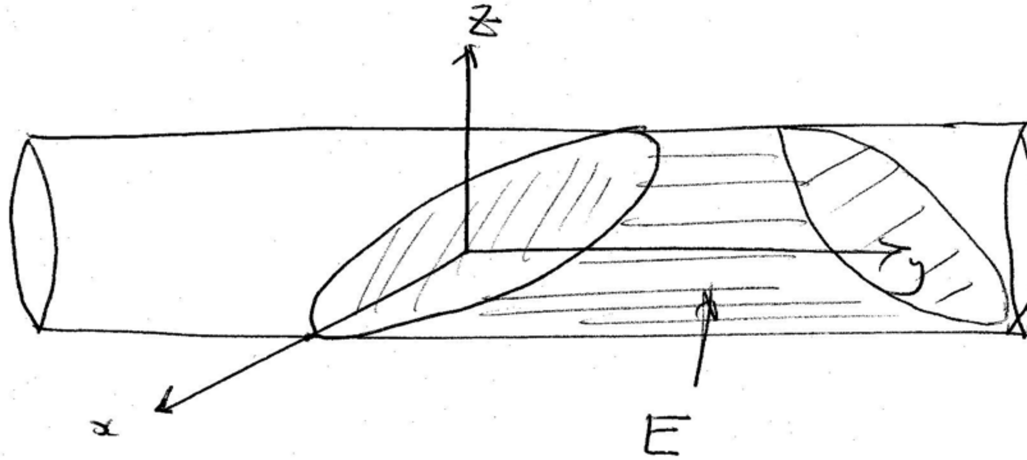
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F})\, dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{2} (\rho^2 \cos^2\phi)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

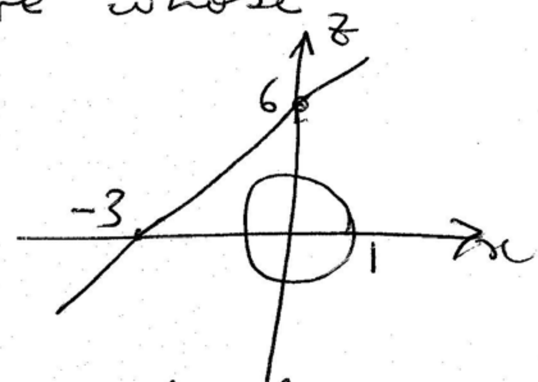
$$= 2\pi \left(\int_0^{\pi} \cos^2\phi \sin\phi\, d\phi\right)\left(\int_0^2 \rho^4\, d\rho\right)$$

$$= \frac{128\pi}{15}$$

(9) [10 pts] Find the volume of the solid region bounded by the surfaces $x^2 + z^2 = 1$, $2y + z = 8$, and $x + y = 1$.



The two planes meet in a curve whose projection to xz-plane is



$$2x + (8 - z) = 2$$

$$z - 2x = 6$$

So these planes intersect OUTSIDE cylinder.
Region can be filled with match sticks parallel to y axis, going from plane $x + y = 1$ to plane $2y + z = 8$.
The match sticks all lie inside cylinder $x^2 + z^2 = 1$
So use (r, θ, y) = Cyl. Coords aligned with y axis

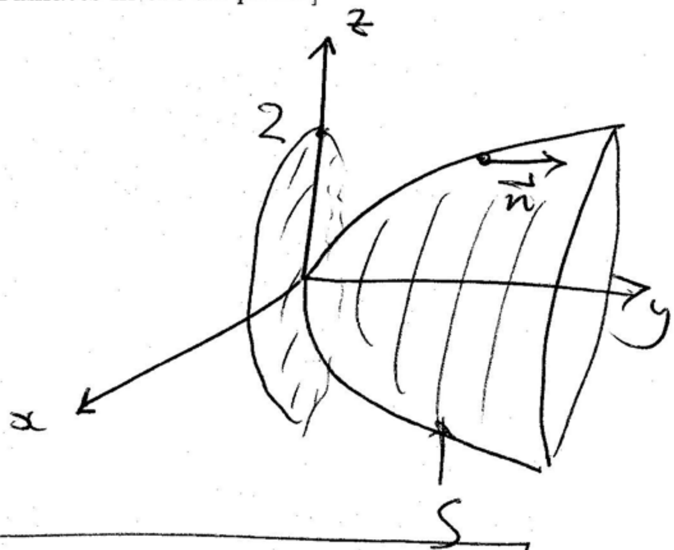
E is $\quad 0 \le r \le 1$

$$0 \le \theta \le 2\pi$$

$$1 - r\cos\theta \le y \le 4 - \frac{r}{2}\sin\theta$$

$$\text{VOL} = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{y=1-r\cos\theta}^{y=4-\frac{r}{2}\sin\theta} 1 .\, r \, dy\, dr\, d\theta$$

(10) [10 pts] Let S be the surface that is the portion of the paraboloid $y = x^2 + z^2$ with $0 \le y \le 4$. We choose the unit normal $\mathbf{n}$ on S to be the one with $\mathbf{n} \cdot \mathbf{j} > 0$. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + z\mathbf{k}$. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$. [Hint: Define parameters for S in terms of polar coordinates in the xz-plane.]



Use parametrization

$x = r\cos\theta$

$\cancel{z} = r\sin\theta$

$y = x^2 + z^2 = r^2$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$\boxed{\vec{x}(r,\theta) = (r\cos\theta, r^2, r\sin\theta)}$$

$$\frac{\partial \vec{x}}{\partial \theta} \times \frac{\partial \vec{x}}{\partial r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & 0 & r\cos\theta \\ \cos\theta & 2r & \sin\theta \end{vmatrix}$$

$$= (-2r^2\cos\theta, \underset{\uparrow}{r}, -2r^2\sin\theta)$$

> 0 ~~so~~ ~~$\vec{x}$~~ IN DIRECTION OF $\vec{n}$.

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (r\cos\theta, 0, r\sin\theta) \cdot (-2r^2\cos\theta, r, -2r^2\sin\theta)\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} -2r^3\, dr\, d\theta = -16\pi$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________