NAME: SOLUTIONS

1 /8	2 /10	3 /12	4 /12	5 /8	T /50

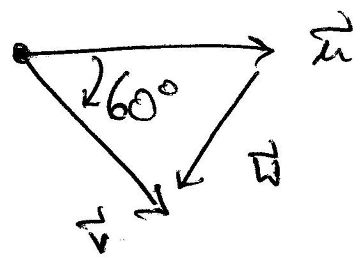
MATH 251 (Fall 2011) Exam I, Sept 29th

No calculators, books or notes! Show all work and give **complete explanations**. This 65 min exam is worth 50 points.

(1) [8 pts] Let **u** be a unit vector in the xy-plane. Think of **u** as a vector that starts at the origin. Let **v** be the vector obtained by rotating **u** clockwise about the origin by 60°. Let $\mathbf{w} = \mathbf{v} - \mathbf{u}$.

(a) Draw a sketch that illustrates how the vectors **u**, **v**, and **w** are related.

$|\vec{u}| = |\vec{v}| = 1$



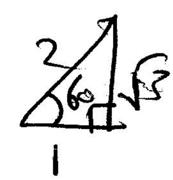
$\vec{w} + \vec{u} = \vec{v}$

EQUILATERAL TRIANGLE

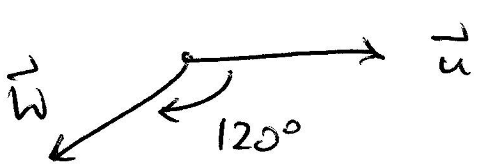
So $|\vec{w}| = 1$

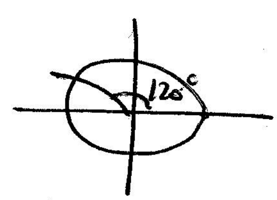
(b) Use the geometric (physics) definitions of the dot product and cross product to find (i) $\mathbf{u} \cdot \mathbf{v}$ (ii) $\mathbf{u} \cdot \mathbf{w}$ and (iii) $\mathbf{u} \times \mathbf{v}$.

(i) $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta = 1 \cdot 1 \cdot \cos 60° = \frac{1}{2}$



(ii) $\vec{u} \cdot \vec{w} = |\vec{u}|\,|\vec{w}| \cos\theta = 1 \cdot 1 \cos 120° = -\frac{1}{2}$





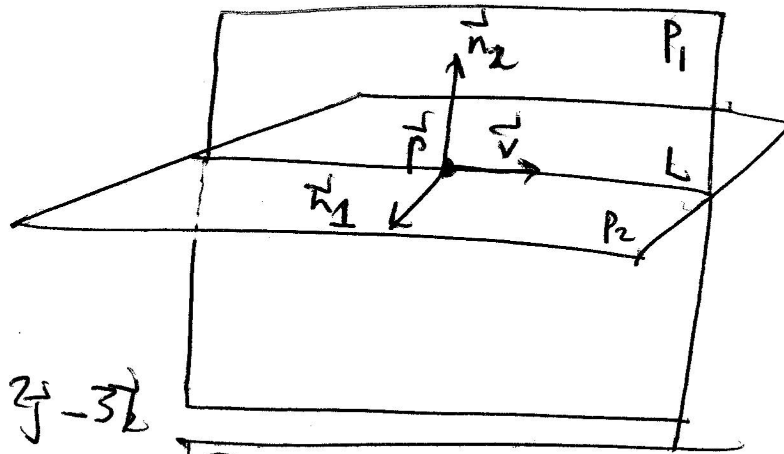
(iii) $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta = 1 \cdot 1 \cdot \sin 60° = \frac{\sqrt{3}}{2}$

Direction of $\vec{u} \times \vec{v}$ is $-\vec{k}$ by RH Rule

So $\vec{u} \times \vec{v} = -\frac{\sqrt{3}}{2}$ ...

(2) [10 pts]

(a) Find a vector parametrization of the line obtained by intersecting the planes $x + 2y + 3z = 1$ and $x - y + z = 2$.



$\vec{r}(t) = \vec{p} + t\vec{v}$

① $\vec{v} = \vec{n}_1 \times \vec{n}_2$ from picture

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 5\vec{i} + 2\vec{j} - 3\vec{k}$$

② Need a point $\vec{p}$ on both planes.

Set $z=0$: $x + 2y = 1$, $x - y = 2$

has solution $x = 5/3$, $y = -1/3$

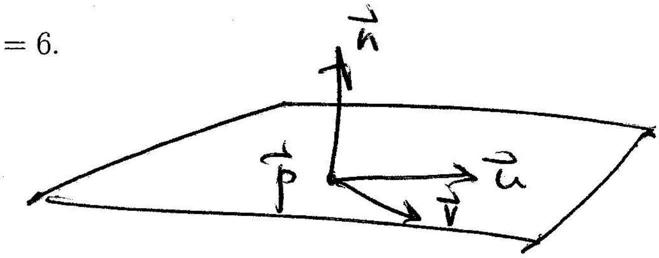
So $\vec{p} = \left(\frac{5}{3}, -\frac{1}{3}, 0\right)$ is on L

③ So

$\vec{r}(t) = \left(\frac{5}{3} + 5t, -\frac{1}{3} + 2t, -3t\right)$

There are other correct solutions too.

(b) Find a vector parametrization of the plane $x + 2y + 3z = 6$.



$\vec{r}(s,t) = \vec{p} + s\vec{u} + t\vec{v}$

① The points

$P = (6,0,0)$

$Q = (0,3,0)$

$R = (0,0,2)$

are on plane

② Set

$\vec{u} = \overrightarrow{PQ} = Q - P = (-6,3,0)$

$\vec{v} = \overrightarrow{PR} = R - P = (-6,0,2)$

$\vec{p} = P = (6,0,0)$

③ So $\vec{r}(s,t) = (6,0,0) + s(-6,3,0) + t(-6,0,2)$

There are other correct solutions too

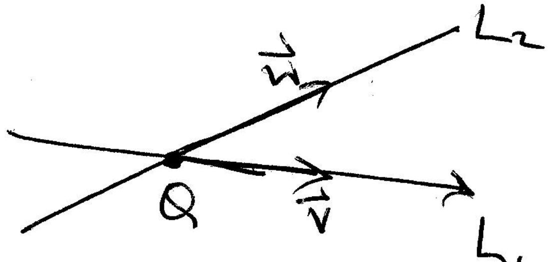
(3) [12 pts] Let L_1 and L_2 be lines in space with parametrizations

$$x = 1 + 2t \qquad y = 2 + t \qquad z = -4 + t$$

and

$$x = 1 + 5t \qquad y = 2 + 2t \qquad z = -4.$$

(a) Using a schematic diagram and an English sentence, explain why L_1 and L_2 lie in a plane, P.



① The point $Q = (1, 2, -4) = \vec{p}$ lies on both lines (set $t=0$).

② The lines are not ∥ as $\vec{v} = (2,1,1)$ is not parallel to $\vec{w}(5,2,0)$

③ Any ~~two~~ 2 ~~points~~ lines intersect in a point and are not ∥, they define a plane.

(b) Find a vector parametrization of the plane P.

$$\vec{r}(s,t) = \vec{p} + s\vec{v} + t\vec{w}$$

$$= (1,2,-4) + s(2,1,1) + t(5,2,0)$$

(c) Find a level set equation of the plane P.

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 5 & 2 & 0 \end{vmatrix} = (-2, 5, -1)$$

$$(\vec{r} - \vec{p}) \cdot \vec{n} = 0$$

$$(x-1, y-2, z+4) \cdot (-2, 5, -1) = 0$$

$$-2(x-1) + 5(y-2) - 1(z+4) = 0$$

$$-2x + 5y - z = 12$$

(4) [12 pts] Find the traces (i.e., slices) of the surface

$$-x^2 + 4y^2 - 9z^2 = 4$$

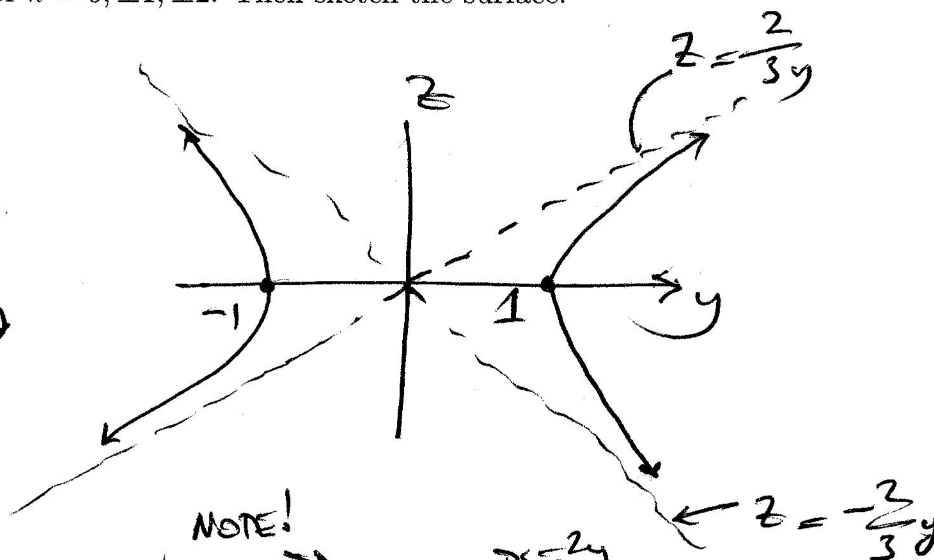
in the planes $x = 0$, $z = 0$, and $y = k$, for $k = 0, \pm 1, \pm 2$. Then sketch the surface.

$x=0$ $4y^2 - 9z^2 = 4.$

Asymptotes $4y^2 = 9z^2$

$z = \pm \frac{2}{3}y$

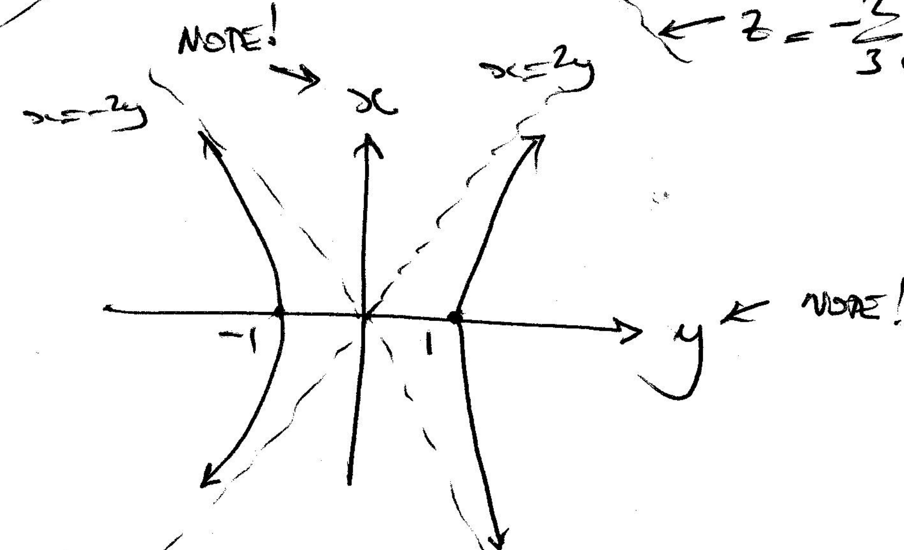
$z=0 \Rightarrow y = \pm 1$



$z=0$ $-x^2 + 4y^2 = 4.$

Asymptotes $x = \pm 2y$

Goes thru $(0, \pm 1) = (x, y)$



$y=0$ $+x^2 + 9z^2 = -4$

EMPTY SET

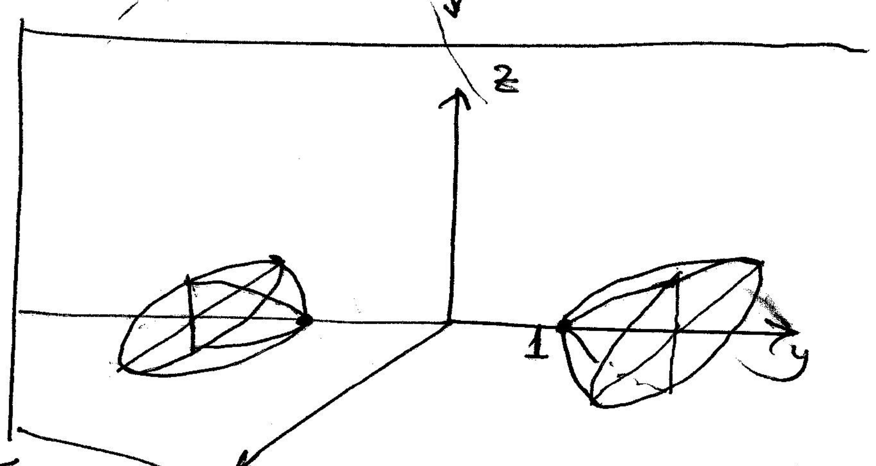
(NO SOLUTIONS)

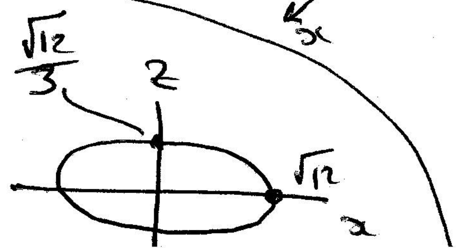
$y = \pm 1$ $x^2 + 9z^2 = 0.$

JUST ORIGIN

$y = \pm 2$ $x^2 + 9z^2 = 12$

$$\left(\frac{x}{\sqrt{12}}\right)^2 + \left(\frac{z}{\frac{\sqrt{12}}{3}}\right)^2 = 1$$





(5) [8 pts] Which of the following expressions are meaningful? Which are meaningless? Explain! (Here **a**, **b**, **c**, and **d** are vectors in space.)

(a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$

$\vec{a} \cdot \vec{b}$ is a scalar
$\vec{c}$ is a vector

DOT PRODUCT

Can't take dot product of scalar + vector. So meaningless

(b) $(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}$

$\vec{a} \cdot \vec{b}$ is scalar
$\vec{c}$ is vector

Can't add scalar + vector
So Meaningless

(c) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

$\vec{a} \times \vec{b}$ and $\vec{c}$ are both vectors
Can take their dot products
Has Meaning

(d) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

$\vec{a} \times \vec{b}$, $\vec{c} \times \vec{d}$ are both vectors. Can take dot product
Has Meaning.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________