NAME: SOLUTIONS.

1 /10	2 /10	3 /10	4 /6	5 /8	6 /6	T /50

MATH 251 (Fall 2011) Exam II, Oct 27th

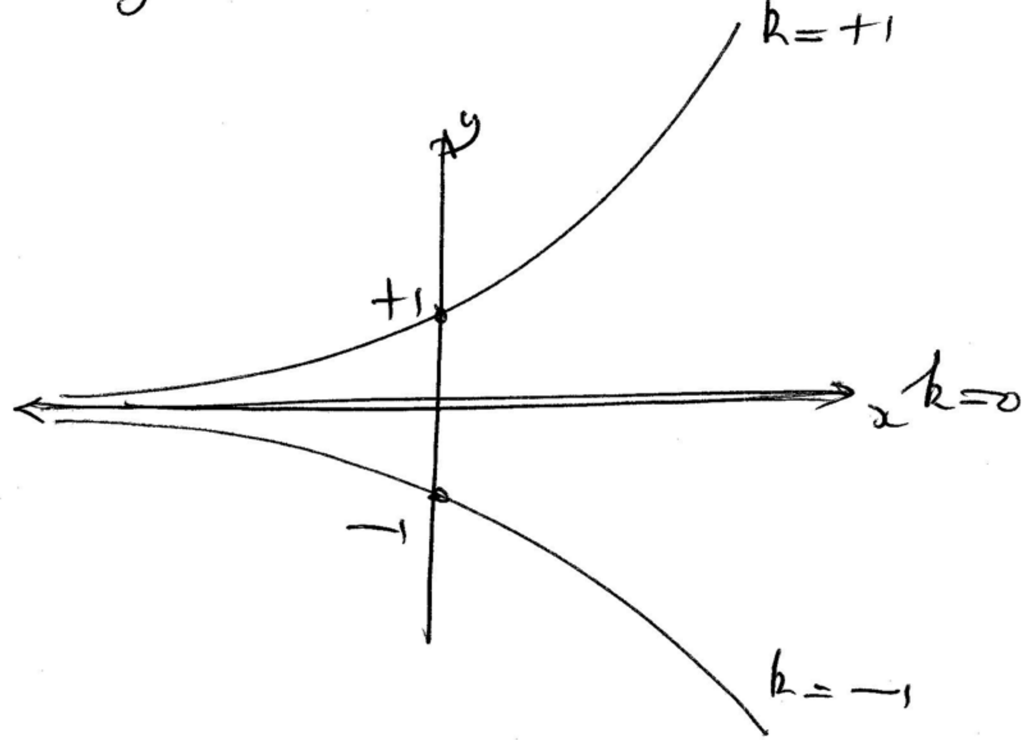
No calculators, books or notes! Show all work and give **complete explanations**. This 65 min exam is worth 50 points.

(1) [10 pts] Sketch the level curves (i.e. contours) of $z = f(x, y) = ye^{-x}$ at levels $k = -1$, 0, and 1.

$z = k \rightsquigarrow k = ye^{-x}$ or $y = ke^{x}$

$k = \pm 1$: $y = \pm e^{x}$

$k = 0$: $y = 0$.



(2) [10 pts] Consider the curve, C, in the plane parametrized by $(x, y) = \mathbf{r}(t) = (2\sin t, \cos t)$ for $0 \le t \le 2\pi$.

(a) Find $\mathbf{r}'(\pi/4)$.

$$\vec{r}\,'(t) = (2\cos t, -\sin t)$$

$$\vec{r}\,'(\pi/4) = \left(\frac{2}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

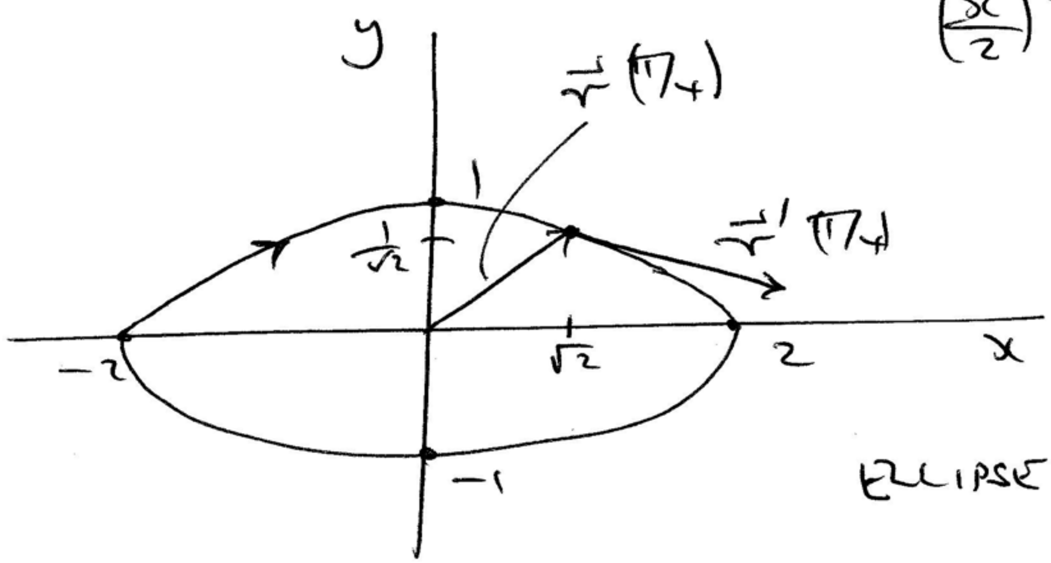
(b) Find a parametrization for the tangent line to the curve, C, at $t = \pi/4$.

$$\vec{l}(s) = \vec{r}(\pi/4) + (s - \pi/4)\,\vec{r}\,'(\pi/4)$$

$$= \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right) + (s - \pi/4)\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$$

(c) Sketch the curve, C, and include in your sketch the vectors $\mathbf{r}(\pi/4)$ and $\mathbf{r}'(\pi/4)$.

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$



ELLIPSE

(3) [10 pts]

(a) Let $z = f(x, y)$ be a function with table of values given by

		y		
		4	5	6
x	1	9	11	14
	2	4	7	9
	3	0	6	8

Estimate $\frac{\partial f}{\partial x}$ at the points $(x, y) = (2, 4)$ and $(2, 5)$. Uses these two estimates to estimate $\frac{\partial^2 f}{\partial y \partial x}$ at $(2, 4)$.

$$\frac{\partial f}{\partial x}(2,4) \simeq \frac{f(3,4) - f(2,4)}{1} = \frac{0-4}{1} = -4$$

$$\frac{\partial f}{\partial x}(2,5) \simeq \frac{f(3,5) - f(2,5)}{1} = \frac{6-7}{1} = -1$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(2,4) \simeq \frac{\frac{\partial f}{\partial x}(2,5) - \frac{\partial f}{\partial x}(2,4)}{1}$$

$$= -1 - (-4) = 3.$$

(b) Calculate the equation of the tangent plane to the graph of the function $f(x, y) = x^2y^3$ at $(x, y) = (2, 1)$.

$$\frac{\partial f}{\partial x} = 2xy^3 = 2 \times 2 \times 1^3 = 4 \text{ @ } (2,1)$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 = 3.4.1 = 12 \text{ @ } (2,1)$$

$$f(2,1) = 4.$$

$$z = f(2,1) + \frac{\partial f}{\partial x}(2,1)(x-2) + \frac{\partial f}{\partial y}(2,1)(y-1)$$

$$= 4 + 4(x-2) + 12(y-1)$$

(4) [6 pts] Either calculate the following limit or prove that it does not exist: $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^3+5y^3}$.

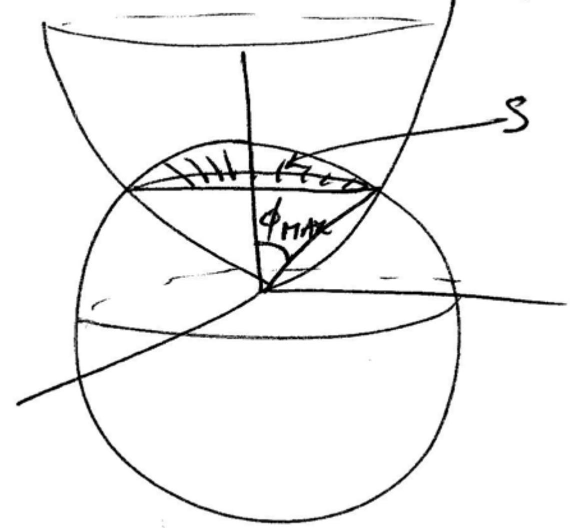
<u>Along $x=0$</u> $\quad \lim_{y\to 0} \frac{0}{0+5y^3} = \lim_{y\to 0} 0 = 0$

<u>Along $y=0$</u> $\quad \lim_{x\to 0} \frac{x^3}{x^3+0} = \lim_{x\to 0} 1 = 1$

Since $0 \neq 1$ limit DNE

(5) [8 pts] Parametrize that part of the surface $x^2+y^2+z^2=4$ that lies above the surface $z=x^2+y^2$.

Surface is cap on sphere, S.
Set $\rho=2$ in spherical coordinates formula to get



$x = 2\sin\phi\cos\theta$
$y = 2\sin\phi\sin\theta$
$z = 2\cos\phi$

<u>Range of θ, ϕ</u>

$0 \le \theta \le 2\pi$ (Rotationally symmetric about z axis)

$0 \le \phi \le \phi_{MAX}$ where ϕ_{MAX} satisfies $z = 2\cos\phi_{MAX}$ and z is given by finding where surfaces meet:

$z + z^2 = 4 \Rightarrow z = \frac{-1+\sqrt{17}}{2}$ by quadratic formula

(6) [6 pts] If $\mathbf{r}(t) \neq \mathbf{0}$, show that

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t)\cdot\mathbf{r}'(t).$$

Hint: $|\mathbf{r}(t)|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$.

Take $\frac{d}{dt}$ of $|r(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$

$$\frac{d}{dt}\left(|r(t)|^2\right) = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right)$$

By CRule from Calc I + Product Rule for dot product:

$$2|r(t)|\cdot\frac{d}{dt}|r(t)| = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$$

$$= 2\,\vec{r}(t)\cdot\vec{r}\,'(t)$$

So $$\frac{d}{dt}|r(t)| = \frac{1}{|r(t)|}\,\vec{r}(t)\cdot\vec{r}\,'(t)$$

<u>NOTE</u> Can also use $|r(t)| = \sqrt{\vec{r}(t)\cdot\vec{r}(t)}$ and take d/dt of both sides, but I hate square roots.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________