NAME: SOLUTIONS

1 /8	2 /10	3 /10	4 /12	5 /10	T /50

MATH 251 (Fall 2011) Exam III, Nov 22nd

No calculators, books or notes! Show all work and give **complete explanations**. This 65 min exam is worth 50 points.

(1) [8 pts] Let C be the straight line segment in the xy-plane from the point $(1,2)$ to the point $(5,3)$. Let $\mathbf{F}$ be the vector field in the plane defined by $\mathbf{F}(x,y) = \frac{1}{2}(x\mathbf{i} + y\mathbf{j})$.

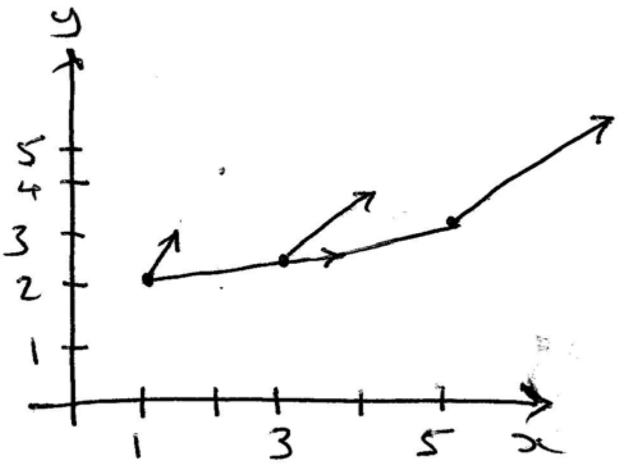
(a) Make a sketch showing the vector $\mathbf{F}(x,y)$ at three points (x,y) on C. Using your sketch, determine whether $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero. Explain!

The angle between $\vec{F}$ and the tangent vector $\vec{T}$ to C is always acute. So

$\vec{F}\cdot\vec{T} = |\vec{F}|\,|\vec{T}|\cos\theta > 0$

(as $-\pi/2 < \theta < \pi/2$)

So $\int_C \vec{F}\cdot d\vec{r} = \int_C (\vec{F}\cdot\vec{T})\,ds > 0$.



(b) Now calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\vec{r}(t) = (1,2) + t\left((5,3)-(1,2)\right) = (1,2) + t(4,1)$$
$$= (1+4t,\ 2+t) \qquad 0 \le t \le 1$$

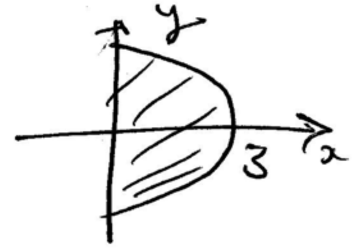
$$\vec{r}\,'(t) = (4,1)$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{1}{2}\int_0^1 (1+4t,\ 2+t)\cdot(4,1)\,dt$$

$$= \frac{1}{2}\int_0^1 6 + 17t\ dt = \frac{29}{4}$$

(2) [10 pts]

(a) Let D be the half-disc in the xy-plane given by $x^2 + y^2 \leq 9$ and $x \geq 0$. Calculate $\iint_D e^{-(x^2+y^2)}\, dA$.



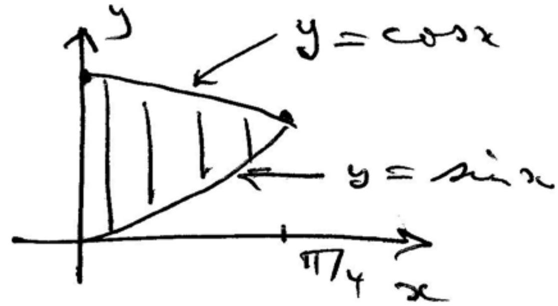
$$\iint_D e^{-(x^2+y^2)} dA$$

$$= \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{3} e^{-r^2} r\, dr\, d\theta$$

$$= \left(\int_{-\pi/2}^{\pi/2} 1\, d\theta\right) \int_{r=0}^{3} e^{-r^2} r\, dr \qquad u = r^2,\ du = 2r\,dr$$

$$= \frac{\pi}{2} \int_{u=0}^{u=9} e^{-u}\, du = \frac{\pi}{2}\left[-e^{-u}\right]_0^9 = \frac{\pi}{2}\left(1 - e^{-9}\right)$$

(b) Let D be the region in the *first quadrant* (*i.e.*, $x \geq 0$ and $y \geq 0$) of the xy-plane that is bounded by the y axis and the curves $y = \sin x$ and $y = \cos x$, and such that $x \leq \pi/4$. Calculate $\iint_D y\, dA$.



$$0 \leq x \leq \frac{\pi}{4}$$

$$\sin x \leq y \leq \cos x$$

$$\iint_D y\, dA$$

$$= \int_{x=0}^{\pi/4} \int_{y=\sin x}^{y=\cos x} y\, dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{\pi/4} \left[y^2\right]_{y=\sin x}^{y=\cos x} dx$$

$$= \frac{1}{2} \int_0^{\pi/4} \cos^2 x - \sin^2 x\, dx = \frac{1}{2} \int_0^{\pi/4} \cos 2x\, dx$$

$$= \frac{1}{4}\left[\sin 2x\right]_0^{\pi/4} = \frac{1}{4}$$

(3) [10 pts] Let $\mathbf{r}(t) = (2\cos t, 3\sin t)$, for $0 \le t \le 2\pi$, and let $(u,v) = F(x,y) = (3x+2y, x^2+5y^2)$. The composition $\mathbf{s}(t) = F(\mathbf{r}(t))$ is a curve in the plane. Use the *Chain Rule from Multivariable Calculus* to answer the following two questions.

(a) At which times, t, is the tangent vector to the curve $(u,v) = \mathbf{s}(t)$ vertical?

T.V. is vertical when $s'(t) = u'(t)\vec{i} + v'(t)\vec{j} = \alpha\vec{j}$

for some scalar α., ie when $\boxed{u'(t) = 0}$.

Now by Chain Rule

$$0 = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 3(-2\sin t) + 2(3\cos t)$$

$$0 = 6(\cos t - \sin t)$$

So $\cos t = \sin t$ ie $\tan t = 1$, ie $t = \frac{\pi}{4}, \frac{5\pi}{4}$

(b) For each of the times you found in (a), is the tangent vector pointing in the $+\mathbf{j}$ or $-\mathbf{j}$ direction?

T.V in $+\vec{j}$ direction $\iff \frac{dv}{dt} > 0$

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}$$

$$= 2(2\cos t)(-2\sin t) + 10(3\sin t)3\cos t$$

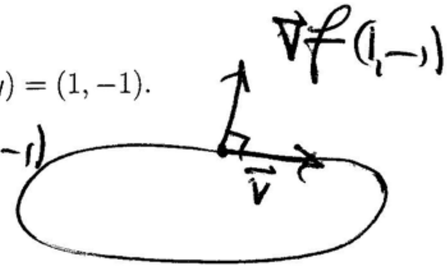
$$= 82\cos t\sin t$$

At $t = \pi/4$ $\quad \frac{dv}{dt} = 82\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} > 0 \quad \boxed{+\vec{j}}$

$t = \frac{5\pi}{4}$ $\quad \frac{dv}{dt} = 82\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) > 0 \quad \boxed{+\vec{j}}$

(4) [12 pts] Let $z = f(x, y) = x^3 - 12xy + 8y^3$.

(a) Find a tangent vector to the level curve $f(x, y) = 5$ at the point $(x, y) = (1, -1)$.



A tangent vector $\vec{v}$ to $f(x,y) = 5$ at $(1,-1)$ must be $\perp$ to $\nabla f(1,-1)$.

Now $\nabla f = (3x^2 - 12y, -12x + 24y^2)$

$\nabla f(1,-1) = (15, 12)$

$(\vec{v} \cdot \nabla f(1,-1) = 0)$

So choose $\vec{v} = (12, -15)$ for example

(b) Find all local maxima, local minima, and saddle points of f.

CPTS $0 = 3x^2 - 12y$ $\quad \Leftrightarrow \quad$ $x^2 = 4y$ ①

$0 = -12x + 24y^2$ $\quad$ $x = 2y^2$ ②

So by ①, ②

$4y^4 = x^2 = 4y$

$\Leftrightarrow y(y^3 - 1) = 0$

$\Leftrightarrow y = 0$ or $y^3 = 1$

$\Leftrightarrow y = 0$ or $y = +1$.

Now $y = 0 \Rightarrow x = 0$ by ③

$y = 1 \Rightarrow x = 2$ by ②

CPTS $(0,0)$, $(2,1)$

$D = \begin{vmatrix} 6x & -12 \\ -12 & 48y \end{vmatrix}$

$\boxed{(0,0)}$

$D = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -144 < 0$

Saddle Point

$\boxed{(2,1)}$ $D = \begin{vmatrix} 12 & -12 \\ -12 & 48 \end{vmatrix}$

$= 12 \times 48 - 12 \times 12 > 0$

$f_{xx} = 12 > 0$

Local Min.

(5) [10 pts] Let $z = f(x, y)$ be a function such that

(x,y)	$(2,1)$	$(-2,-1)$	$(0,\sqrt{3})$	$(\sqrt{3},0)$
$\frac{\partial f}{\partial x}$	-10	10	0	4
$\frac{\partial f}{\partial y}$	-2	4	0	-3

Which of the (x, y) values in this table are candidates for the absolute maximum and absolute minimum of f on the curve $2x^2 - 3xy + 4y^2 = 6$? *Carefully justify your answers!*

This is a constrained optimization problem.
So candidates are solutions of Lagrange multiplier eqns:

$$\boxed{\begin{array}{l} \nabla f = \lambda \nabla g \\ g = c \end{array}}$$

where $g(x,y) = 2x^2 - 3xy + 4y^2$
$c = 6$.

Now $\frac{\partial g}{\partial x} = 4x - 3y$

$\frac{\partial g}{\partial y} = -3x + 8y$

	$(2,1)$	$(-2,-1)$	$(0,\sqrt{3})$	$(\sqrt{3},0)$
$\frac{\partial g}{\partial x}$	5	-5	$-3\sqrt{3}$	$4\sqrt{3}$
$\frac{\partial g}{\partial y}$	2	-2	$8\sqrt{3}$	$-3\sqrt{3}$
g	6 ✓	6 ✓	12 X	6 ✓
$\nabla f = \lambda \nabla g$?	X	✓	—	✓
λ		-2		$\sqrt{3}$

Pledge: *I have neither given nor received aid on this exam*

Signature: ________________

$\boxed{\text{So candidates are } (-2,-1) \text{ and } (\sqrt{3},0)}$