NAME:

1 /12	2 /18	3 /17	4 /18	5 /10	T /75

MATH 251H (Fall 2006) Exam 1, Sept 27th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [~~12~~ 15 pts] Suppose that

$$\mathbf{r}(s,t) \;=\; (1+2s-3t, 5+s, -3+4s-t)$$

is a parametrization of a plane. Find a level set equation for this plane, *i.e.*, an equation of the form

$$ax+by+cz=d.$$

Writing

$$\vec{r}(s,t) = \vec{r}_0 + s\vec{u} + t\vec{v}$$

we have $\vec{r}_0 = (1,5,-3)$, a point in the plane

$\vec{u} = (2,1,4)$

and $\vec{v} = (-3,0,-1)$ two vectors that lie in the plane. $\vec{n} = \vec{u} \times \vec{v}$ is therefore a normal vector to the plane and the level set equation of the plane is $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$ where now $\vec{r} = (x,y,z)$

We have

$$\vec{n} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & 4 \\ -3 & 0 & -1 \end{vmatrix} = -\vec{\imath} - (-2+12)\vec{\jmath} + 3\vec{k} = (-1,-10,3)$$

So eqn is $(x-1, y-5, z+3) \cdot (-1,-10,3) = 0$

or $-(x-1) - 10(y-5) + 3(z+3) = 0$, $\boxed{-x - 10y + 3z + 60 = 0}$

(2) [~~18~~ 15 pts] Consider the parametrized curve $\mathbf{r}(t) = t\mathbf{i} + \frac{\sqrt{2}}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$.
(a) Find a parametrization for the tangent line to this curve at $t = 1$.

$\vec{r}\,'(t) = (1, \sqrt{2}\,t, t^2)$

If give $\vec{r}\,'(1)$ Get 3

$\vec{r}(1) = (1, \frac{\sqrt{2}}{2}, \frac{1}{3})$ 1

$r'(1) = (1, \sqrt{2}, 1)$ 3

$\vec{l}(s) = \vec{r}(1) + s\,\vec{r}\,'(1)$ 3

$= (1+s, \sqrt{2}/2 + \sqrt{2}\,s, \frac{1}{3} + s)$

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(b) Calculate the arc-length function of the curve $\mathbf{r}$.

$s(t) = \int_0^t |r'(u)|\,du$ 3

$|\vec{r}\,'(u)|^2 = 1 + 2u^2 + u^4 = (1+u^2)^2$

So $|r'(u)| = 1+u^2$ 3

$s(t) = \int_0^t (1+u^2)\,du = \left[u + \frac{1}{3}u^3\right]_0^t = t + \frac{1}{3}t^3$ 2

8

omit

(c) Calculate the curvature of the curve.

$\cancel{K(t) = \frac{T'(t)}{|r'(t)|}}$

$\cancel{T(t) = \frac{r'(t)}{|r'(t)|} = \frac{(1, \sqrt{2}t, t^2)}{1+t^2}}$

$\cancel{T(t) = \left(\frac{1}{1+t^2}, \frac{\sqrt{2}t}{1+t^2}, \frac{t^2}{1+t^2}\right)}$

(3) [17 pts] Show that the parametrized curve $\mathbf{r}(t) = (\cos t, \sin t, 1)$ lies on the following two surfaces:

(i) $\rho = \sqrt{2}$ (in spherical coordinates)
(ii) $z = r$ (in cylindrical coordinates).

Also sketch both surfaces and the curve in the same figure.

(i) $\rho = \sqrt{2}$ is $x^2+y^2+z^2 = \rho^2 = 2$ is a sphere center $(0,0,0)$ radius $\sqrt{2}$. 2

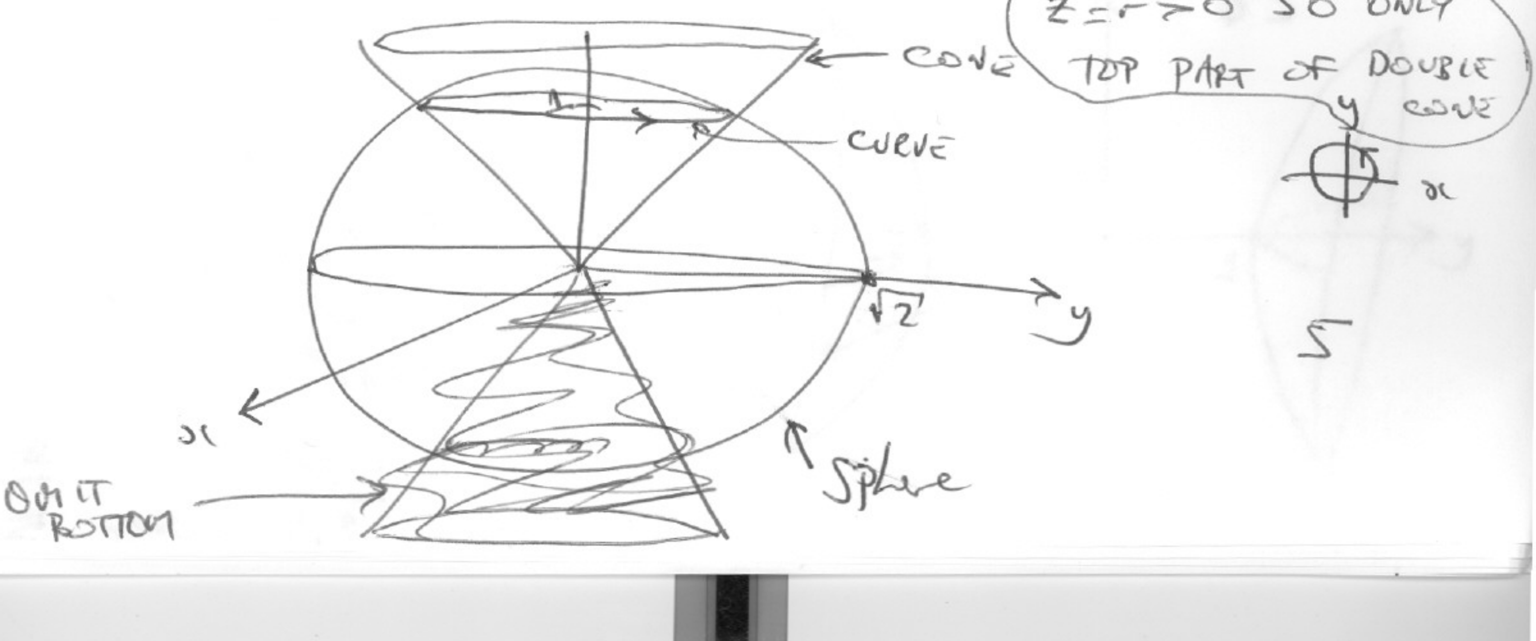
$x^2(t) + y^2(t) + z^2(t) = \cos^2 t + \sin^2 t + 1^2 = 2$ shows that $\vec{r}(t)$ lies on $\rho = \sqrt{2}$ 4

(ii) $z = r$ is $z = \sqrt{x^2+y^2}$ or $z^2 = x^2+y^2$, $z \geq 0$ which is a cone. Now 2

$z^2(t) = 1^2$

$x^2(t) + y^2(t) = \cos^2 t + \sin^2 t = 1$ 4

Shows that $z^2(t) = x^2(t) + y^2(t)$ so $\vec{r}$ also lies on the cone.



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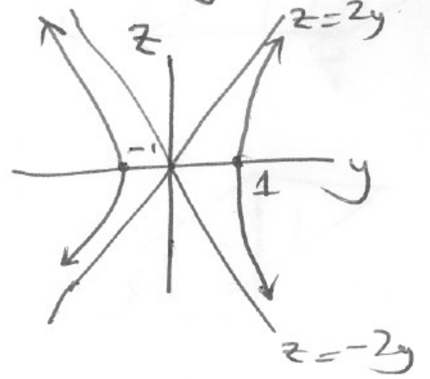
(4) [18 pts] Find the traces (*i.e.*, slices) of the surface

$$-x^2 + 4y^2 - z^2 = 4$$

in the planes $x = 0$, $z = 0$, and $y = k$ for several well-chosen values of k. $= 0, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3$
Also sketch the surface and name it.

$\boxed{x=0}$ $4y^2 - z^2 = 4$

$\boxed{z=0}$ $-x^2 + 4y^2 = 4$



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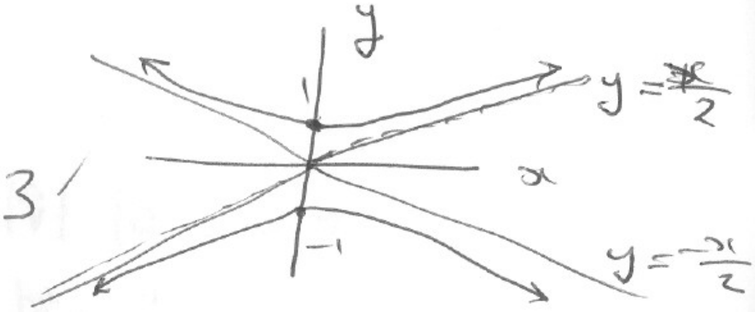

$z = \pm 2y$ are asymptotes

$z = 0 \Rightarrow y = \pm 1$

$x^2 = 4y^2$ or $x = \pm 2y$ are asymptotes

$x = 0 \Rightarrow y = \pm 1$

$\boxed{y=k}$

$x^2 + z^2 = 4(k^2 - 1)$

If $|k| < 1$ Trace is empty

If $k = \pm 1$, Trace is point $(0,0)$

If $|k| > 1$ Trace is circle

radius $2\sqrt{k^2-1}$

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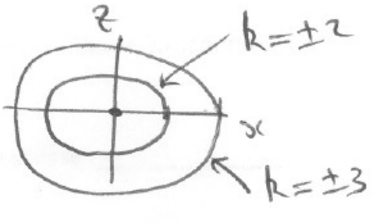

$k=\pm 2$ radius $= 2\sqrt{3}$

$k=\pm 3$ radius $= 2\sqrt{8}$

Elliptic Hyperboloid of 2 Sheets 1
~~are~~ whose axis of symmetry is y axis

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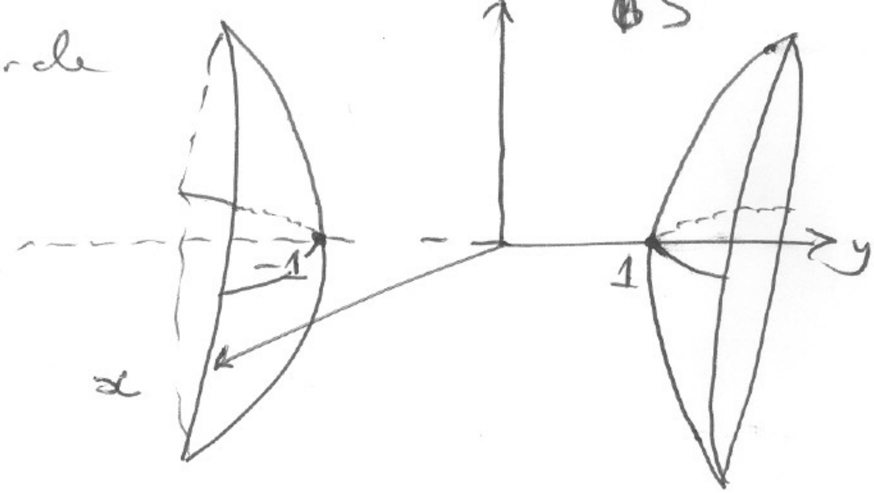
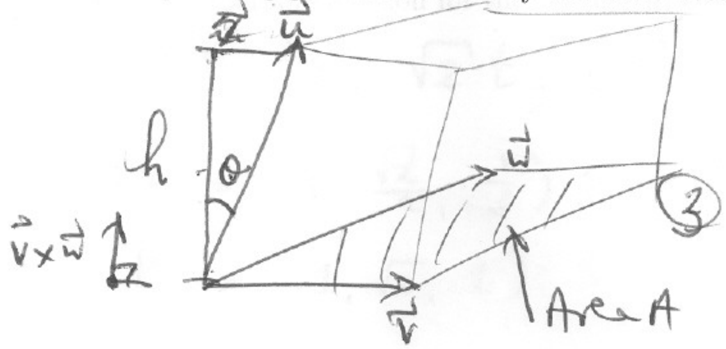
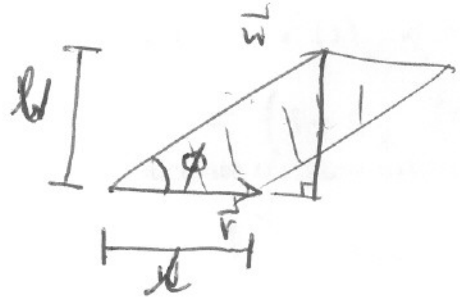


(5) [10 pts] Use the geometric definitions of the dot product and the cross product to show that the volume of the parallelipiped determined by the three vectors **u**, **v** and **w** is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.



$$V = A\,h \quad 1$$



$$A = lw = |\vec{v}|\,|\vec{w}| \sin\phi$$

$$A = |\vec{v} \times \vec{w}| \quad 3$$

$$h = |\vec{u}|\,|\cos\theta| = \frac{|\vec{u}|\,|\vec{u} \cdot (\vec{v} \times \vec{w})|}{|\vec{u}|\,|\vec{v} \times \vec{w}|} = \frac{|\vec{u} \cdot (\vec{v} \times \vec{w})|}{|\vec{v} \times \vec{w}|}$$

So $V = A \cdot h = |\vec{u} \cdot (\vec{v} \times \vec{w})|$. 3

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________