NAME:	SOLUTIONS

1 /15	2 /20	3 /15	4 /8	5 /10	6 /7	T /75

MATH 251H (Fall 2006) Exam 2, Oct 27th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [15 pts] Either compute the following limits or show they do not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-3y^2}{4x^2+5y^2}$

As go to (0,0) along $x=0$ have

$$\lim_{y\to 0} \frac{-3y^2}{5y^2} = \lim_{y\to 0} -\frac{3}{5} = -\frac{3}{5}$$

whereas as go to (0,0) along $y=0$ have

$$\lim_{x\to 0} \frac{x^2}{4x^2} = \lim_{x\to 0} \frac{1}{4} = 1/4.$$

Since these two limits ~~do~~ are not equal, $\lim_{(x,y)\to(0,0)} \frac{x^2-3y^2}{4x^2+5y^2}$ DNE.

(b) $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = L$

OBSERVATION Degree of Numerator = 3 > Deg of Denominator = 2
This suggests limit exists and is 0. To prove that we convert to polar coords using fact that

$(x,y)\to(0,0) \iff r\to 0.$

So $L = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0$

as $|r\cos^3\theta| \le r \to 0$ as $r\to 0$. So by Sandwich Theorem $r\cos^3\theta \to 0$ as $r\to 0$ too.

NOTE SHOWING LIMITS ALONG SEVERAL CURVES ARE ALL 0 IS NOT ENOUGH. See book + class notes.

(2) [20 pts] Let f be the function $z = f(x, y) = x^2 - xy + y^2 + 3x$.

(a) Calculate an equation of the form $z = ax + by + c$ for the tangent plane to the surface $z = f(x, y)$ at the point $(x, y, z) = (2, 3, 13)$.

$\nabla f(x, y) = (2x - y + 3, -x + 2y)$

$\nabla f(2, 3) = (4, 4)$

So equation of tangent plane at $(2, 3, 13)$ is

$z = f(2,3) + \nabla f(2,3) \cdot (x-2, y-3)$

$= 13 + (4,4) \cdot (x-2, y-3)$

$= 13 + 4x - 8 + 4y - 12 = 4x + 4y - 7$

(b) Find and classify all critical points of f.

Critical points occur where $\nabla f(x,y) = (0,0)$. So by (a)

① $2x - y + 3 = 0$

② $-x + 2y = 0 \Rightarrow x = 2y$

Plug ② into ① to get $4y - y + 3 = 0 \Rightarrow y = -1$

and so $x = -2$ by ②

So only 1 CPt at $(-2, -1)$.

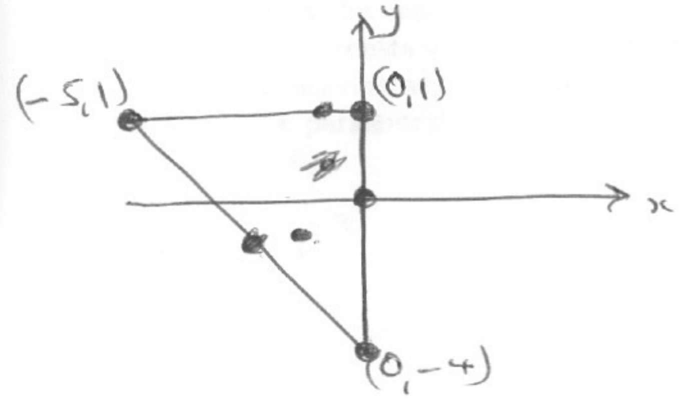
NOTE You should check $\nabla f(-2,-1) = \vec{0}$ holds!

2ND DERIVATIVE TEST

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$$

and $\partial^2 f / \partial x^2 = 2 > 0$ says $(-2, -1)$ is LOCAL MIN

(c) Find the absolute maximum and minimum of f on the region in the xy-plane bounded by the lines $x = 0$, $y = 1$, and $x + y = -4$.



From (b) we have 1 CPt inside D at $(-2,-1)$

Find extrema of f on ∂D.

$x=0$ $\quad -4 \le y \le 1$

$g(y) = f(0,y) = y^2$ has 1 CPt at $y=0$. $(0,0)$

Endpts are $(0,-4)$ and $(0,1)$

$y=1$ $\quad -5 \le x \le 0$

$h(x) = f(x,1) = x^2 + 2x + 1$

$0 = h'(x) = 2x+2 \Rightarrow x = -1,\ y = +1$ $\quad (-1,1)$

Endpts are $(-5,1)$ and $(0,1)$

$x+y=-4$

Set $y = -4-x, \quad -5 \le x \le 0$.

$G(x) = f(x,-4-x) = 3x^2 + 15x + 16$

$0 = G'(x) = 6x + 15 \Rightarrow x = -5/2,\ y = -3/2$ $\quad (-5/2,-3/2)$

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7 CRITICAL POINTS.

$f(-2,-1) = -3$, $f(0,0) = 0$, $f(0,1) = 1$, $f(0,-4) = 16$, $f(-5,1) = 16$

ABS MIN $\quad f(-1,1) = 0 \quad f(-5/2,-3/2) = -1/4$. $\quad$ ABS MAX

(3) [15 pts]
(a) Use equations to explain why $\mathbf{r}(u,v) = (u\cos v, u\sin v, u)$, where $u \geq 0$ and $0 \leq v \leq 2\pi$, is a parametrization of a cone.
(b) Plot the $u =$ constant and $v =$ constant grid curves on a picture of the cone and label them.
(c) Carefully describe what happens to the parametrization and the surface when $u = 0$.
(d) Using the parametrization in (a), calculate a parametrization of the tangent plane to the cone at $(u, v) = (2, \pi/6)$.

(a) $z^2 = u^2 = u^2\cos^2 v + u^2\sin^2 v = x^2+y^2$ is equation of cone. So for each (u,v), $\vec{r}(u,v)$ lies on cone.
Also $x^2+y^2 = u^2$ and $z = u$ say we have a circle of radius u at height u, which means surface is a cone.

(b)

(c) The $v=c$ grid curves all intersect at $(0,0,0)$.
$\frac{\partial \vec{r}}{\partial v} = \vec{0}$ and $\frac{\partial \vec{r}}{\partial u}$ is NOT defined at $(0,0,0)$.
The surface does not have a tangent plane at the vertex $(0,0,0)$ of the cone.

VIEW FROM A POINT ON Z AXIS

(d) $\frac{\partial \vec{r}}{\partial u} = (\cos v, \sin v, 1) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right)$ at $(u,v) = (2, \pi/6)$

$\frac{\partial \vec{r}}{\partial v} = (-u\sin v, u\cos v, 0) = (-1, \sqrt{3}, 0)$ "

$\vec{r}(2, \pi/6) = (\sqrt{3}, 1, 2)$ So $T(s,t) = (\sqrt{3}, 1, 2) + s\left(\sqrt{3}/2, \frac{1}{2}, 1\right) + t(-1, \sqrt{3}, 0)$

(4) [8 pts] Suppose that the directional derivative of a function $w = f(x, y, z)$ at a point P is greatest in the direction of the vector $\mathbf{v} = (1, 1, -1)$, and that in this direction the value of the directional derivative is $2\sqrt{3}$.

(a) What is ∇f at P, and why?

We know ∇f is in direction of $\vec{v}$ so

$$\frac{\nabla f}{|\nabla f|} = \frac{\vec{v}}{|\vec{v}|} \Rightarrow \nabla f = \frac{|\nabla f|}{|\vec{v}|} \cdot \vec{v}.$$

Since $|\nabla f| = 2\sqrt{3}$ we have

$$\nabla f = \frac{2\sqrt{3}}{\sqrt{3}}(1,1,-1) = (2,2,-2)$$

IDEA HERE: To determine a vector (∇f) just need its magnitude + direction cf #13

(b) What is the directional derivative of f in the direction of the vector $(1, 1, 0)$?

$$\vec{u} = \frac{(1,1,0)}{|(1,1,0)|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$D_{\vec{u}} f = \vec{u} \cdot \nabla f$$

$$D_{\vec{u}} f = \frac{1}{\sqrt{2}}(1,1,0) \cdot (2,2,-2) = 2\sqrt{2}$$

(5) [10 pts] Suppose that $g(t) = f(\mathbf{r}(t))$, where $\mathbf{r}$ is the curve $\mathbf{r}(t) = (\cos t, \sin t, t)$ and

$$\frac{\partial f}{\partial x} = x \qquad \frac{\partial f}{\partial y} = y \qquad \frac{\partial f}{\partial z} = z - 2.$$

Find any local maxima and minima of g. (Do *not* find a formula for f.)

Since $g(t) = f(\vec{r}(t))$ the Chain Rule gives

$g: \mathbb{R} \to \mathbb{R}$ So $g'(t)$ is a SCALAR

$$g'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= (\cos t, \sin t, t-2) \cdot (-\sin t, \cos t, 1)$$

$$= -\cos t \sin t + \sin t \cos t + t - 2$$

$$\boxed{g'(t) = t - 2}$$

So $t = 2$ is only CPt of g.

And since $g''(t) = 1 > 0$ This CPt must be a LOCAL MIN

(6) [7 pts] If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, use the Chain Rule to prove that

$$\frac{\partial z}{\partial \theta} = x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}$$

and

$$r\frac{\partial z}{\partial r} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta$$

$$= -\frac{\partial z}{\partial x} y + \frac{\partial z}{\partial y} x \quad \checkmark$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{r}$$

$$= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$= \frac{\partial z}{\partial x}\frac{x}{r} + \frac{\partial z}{\partial y}\frac{y}{r}$$

So

$$r\frac{\partial z}{\partial r} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}.$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________