NAME: SOLUTIONS

1 /10	2 /12	3 /15	4 /10	5 /14	6 /14	T /75

MATH 251H (Fall 2006) Exam 3, Nov 22nd

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 75 minute exam. It is worth a total of 75 points.

(1) [10 pts]
(a) Find the divergence of the vector field $\mathbf{F}(x, y, z) = e^x \sin y\, \mathbf{i} + e^x \cos y\, \mathbf{j} + z\, \mathbf{k}$.

$$\operatorname{DIV}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y) + \frac{\partial}{\partial z}(z)$$

$$= e^x \sin y + - e^x \sin y + 1$$

$$= 1$$

(b) Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y) = x^2 \cos(y)\, \mathbf{i} + y \sin(x)\, \mathbf{j}$ and let C be the curve in the plane given by $y = x^3$ from $(0,0)$ to $(2,8)$. Find a formula for a function g so that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 g(t)\, dt$.

Let $\vec{r}(t) = (t, t^3) \quad 0 \le t \le 2$ parametrize C.

Then $\qquad \vec{r}\,'(t) = (1, 3t^2)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^2 (t^2 \cos(t^3)\vec{i} + t^3 \sin t\, \vec{j}) \cdot (1\vec{i} + 3t^2 \vec{j})\, dt$$

$$= \int_0^2 t^2 \cos(t^3) + 3t^5 \sin t\, dt$$

So

$$g(t) = t^2 \cos(t^3) + 3t^5 \sin t$$

(2) [12 pts] Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = (2xy - \sin x \sin y + e^x)\,\mathbf{i} + (x^2 + \cos x \cos y)\,\mathbf{j}$ and where C is any curve from $(0, 0)$ to $(2, 3)$.

$$P = 2xy - \sin x \sin y + e^x$$

$$Q = x^2 + \cos x \cos y$$

$$\frac{\partial Q}{\partial x} = 2x + -\sin x \cos y$$

$$\frac{\partial P}{\partial y} = 2x - \sin x \cos y$$

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ and $\vec{F}$ is defined on all of $\mathbb{R}^2$,

$\vec{F}$ must be conservative, $\vec{F} = \nabla f$.

We must have

$$\frac{\partial f}{\partial x} = P = 2xy - \sin x \sin y + e^x$$

$$\Rightarrow f(x,y) = \int (2xy - \sin x \sin y + e^x)\, dx + g(y)$$

$$= x^2 y + \cos x \sin y + e^x + g(y) \quad (1)$$

AND

$$\frac{\partial f}{\partial y} = Q = x^2 + \cos x \cos y$$

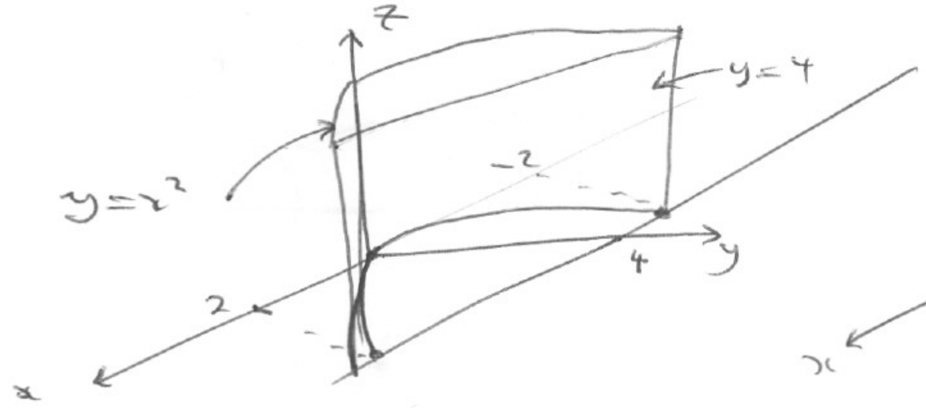
$$\Rightarrow f(x,y) = \int (x^2 + \cos x \cos y)\, dy + h(x)$$

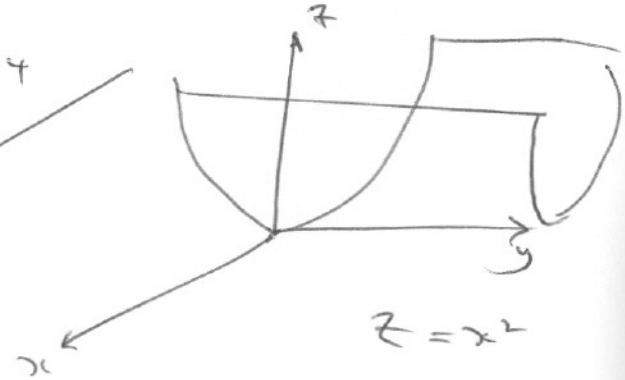
$$= x^2 y + \cos x \sin y + h(x) \quad (2)$$

By (1) + (2) we get $\boxed{f(x,y) = x^2 y + \cos x \sin y + e^x}$

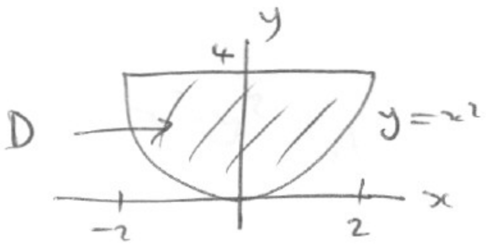
By FTC $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,3) - f(0,0) = 4 \times 3 + \cos 2 \sin 3 + e^2 - 1$

(3) [15 pts] Calculate the volume of the solid enclosed by the parabolic cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$ and $y = 4$.





So we want volume between $z=0$ and $z=x^2$ over the region D given by



$$V = \iint_D (x^2 - 0)\, dA$$

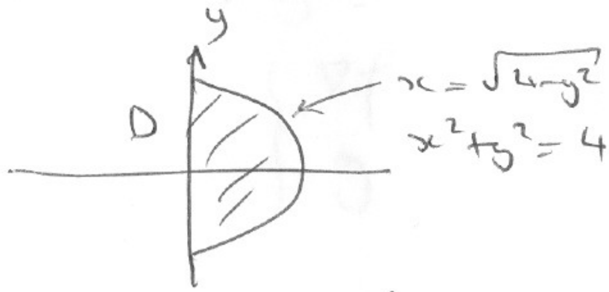
$$= \int_{x=-2}^{2} \int_{y=x^2}^{y=4} x^2 \, dy\, dx$$

$$= \int_{x=-2}^{2} x^2 (4 - x^2)\, dx$$

$$= \int_{-2}^{2} 4x^2 - x^4 \, dx = 2\left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2$$

$$= 2\left(\frac{4}{3}2^3 - \frac{2^5}{5}\right) = 2^4\left(\frac{4}{3} - \frac{2}{5}\right) = 2^4 \times \frac{14}{15}.$$

(4) [10 pts] Calculate the integral $\iint_D e^{-x^2-y^2}\,dA$, where D is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the y-axis.



In polar coordinates D is

$$0 \le r \le 2$$
$$-\pi/2 \le \theta \le \pi/2$$

So

$$\iint_D e^{-x^2-y^2}\,dA = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2} e^{-r^2}\, r\,dr\,d\theta$$

$$= \pi \int_0^2 e^{-r^2}\, r\,dr \qquad u = r^2, \quad du = 2r\,dr$$

$$= \frac{\pi}{2} \int_0^4 e^{-u}\,du$$

$$= \pi/2 \left[-e^{-u}\right]_0^4 = \pi/2\left(1 - e^{-4}\right)$$

(5) [14 pts] Use the Method of Lagrange Multipliers to find the absolute maximum of the function $f(x, y) = (x - y)^3$ subject to the constraint $x^2 + y^2 = 1$.

$g(x,y) = x^2+y^2 = 1$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \quad \text{gives}$$

$3(x-y)^2 = 2\lambda x \quad (1)$

$-3(x-y)^2 = 2\lambda y \quad (2)$

$x^2+y^2 = 1 \quad (3)$

So by (1) and (2):

$\lambda x = -\lambda y \Rightarrow \lambda(x+y)=0 \Rightarrow \lambda=0$ or $y=-x$.

$\boxed{\lambda=0}$

So by (1) $x=y$ and so by (3) $2x^2=1$, $x=\pm\frac{1}{\sqrt2}$

$y=\pm\frac{1}{\sqrt2}$

$f(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}) = 0$

$f(-\frac{1}{\sqrt2}, -\frac{1}{\sqrt2}) = 0$

$\boxed{y=-x}$ By (3) $2x^2=1$, $x=\pm\frac{1}{\sqrt2}$, $y=\mp\frac{1}{\sqrt2}$

$$\boxed{f\left(\frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right) = \left(\frac{1}{\sqrt2}+\frac{1}{\sqrt2}\right)^3 = \left(\frac{2}{\sqrt2}\right)^3 = (\sqrt2)^3 = 2^{3/2}}$$ ABS MAX

$f(-\frac{1}{\sqrt2}, +\frac{1}{\sqrt2}) = -2^{3/2}$

[Note If $y=-x$ the (1) (3) gives $\lambda = \frac{3(x-y)^2}{2x} = \frac{3\cdot4x^2}{2x} = 6x = \frac{6}{\sqrt2}$]

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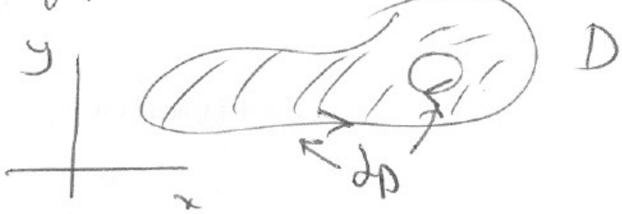
(6) [14 pts] Carefully state Green's Theorem and use it to calculate the integral

$$I = \int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy,$$

where C is the positively-oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Let D be a region in $\mathbb{R}^2$ with boundary curve ∂D, oriented so that if you walk around ∂D with head in $+z$-direction, Then D is on your left:

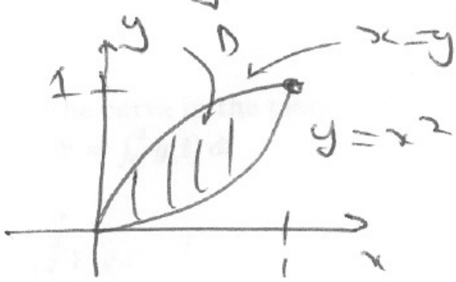
Then if $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a Vector Field on D we have



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy.$$

$$I = \iint_D \left[\frac{\partial}{\partial x}\left(2x + \cos(y^2)\right) + - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$$

$$= \iint_D (2-1)\,dA = \text{Area}(D)$$



$$= \int_{x=0}^{1} \int_{y=x^2}^{y=\sqrt{x}} 1\,dy\,dx$$

$$= \int_{x=0}^{1} (\sqrt{x} - x^2)\,dx = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________