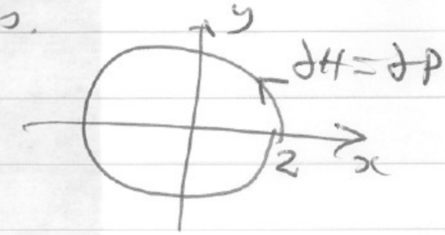
# M251 HWK SOLUTIONS 17.8, 16.7

**17.8 #1**

Orient $H$ and $P$ to both have upward normals.
Then $\partial H = \partial P$ as oriented curves.
So by Stokes' Thm




$$\iint_H (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial H} \vec{F} \cdot d\vec{r}$$

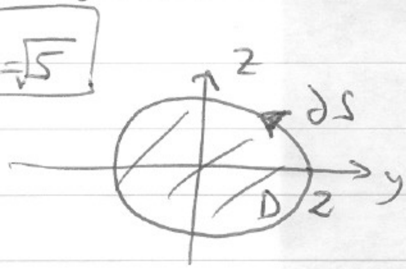
$$= \int_{\partial P} \vec{F} \cdot d\vec{r} = \iint_P (\nabla \times \vec{F}) \cdot d\vec{S}.$$

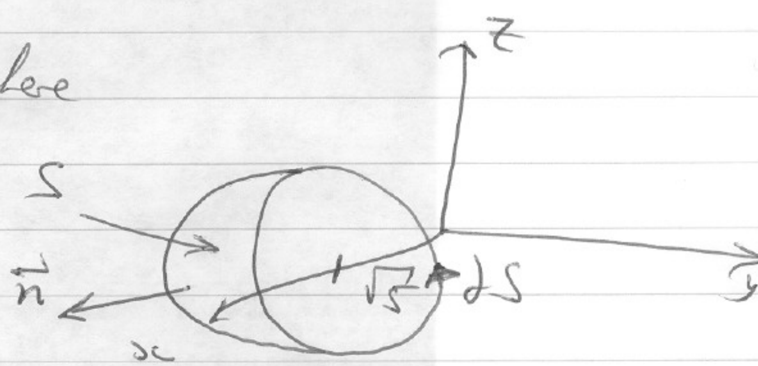
**#4** The Hemisphere and Cylinder intersect in a circle radius 2 in the plane $x = \sqrt{5}$ since

$$x = \sqrt{9 - y^2 - z^2} = \sqrt{9-4} = \sqrt{5} \text{ there.}$$

So $S$ is a "cap" of a sphere
And $\partial S$ is the circle:

$\boxed{x = \sqrt{5}}$




Now $\partial S = \partial D$ where $D$ is the disk in plane $\sqrt{5}$.
with normal $\vec{n} = \vec{i}$.

So as in #1

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}.$$

↑ THIS IS MORE USEFUL

AAH!

$\int_{\partial S} \vec{F} \cdot d\vec{r}$

Parametrize $\partial S$ by

$\vec{r}(t) = (\sqrt{5}, 2\cos t, 2\sin t)$

$0 \le t \le 2\pi$

$= \int_{0}^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$\cancel{\vec{r}\,'(t) = (0, -2\sin t, 2\cos t)}$

$\vec{r}\,'(t) = (0, -2\sin t, 2\cos t)$

$= \int_0^{2\pi} (2\cos t)^2 \, 2\sin t(-2\sin t) + 2\sin t \, 2\cos t \, dt$

$= \int_0^{2\pi} -16\cos^2 t \sin^2 t + 4\cos t \sin t \, dt$

= YOU CAN DO THIS.

$\boxed{\#8}$

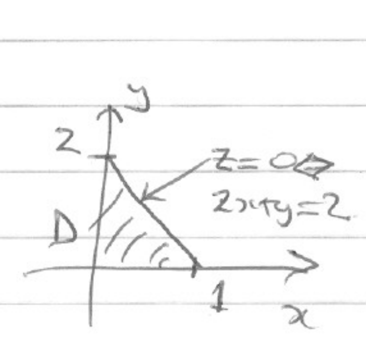
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{-x} & e^{x} & e^{z} \end{vmatrix} = 0\vec{i} + 0\vec{j} + e^{x}\vec{k}$$

$\nabla \times \vec{F} = e^{x}\vec{k}$

Surface S is $2x + y + 2z = 2$

where $x \ge 0$, $y \ge 0$, $z \ge 0$.

$z = \dfrac{2 - 2x - y}{2} = 1 - x - y/2 = f(x,y)$




S is Graph of $z = f(x,y)$ over domain D shown.

Also Normal to $S$ is $\vec{n} = (2, 1, 2)$ UPWARD ✓

So $(\nabla \times \vec{F}) \cdot \vec{n} = 2 e^x$.

And therefore

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_S 2e^x \, dS$$

$$= \iint_D 2e^x \sqrt{1 + 1^2 + \left(\tfrac{1}{2}\right)^2} \, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{2-2x} 2e^x \sqrt{2 + \tfrac{1}{4}} \; \cancel{dy\,dx} \; dy\, dx$$

$$= \frac{2.3}{2.} \int_{y=0}^{2} \int_{x=0}^{x=\frac{y-2}{2}} e^x \, dx\, dy \qquad \text{EASIER THIS WAY!}$$

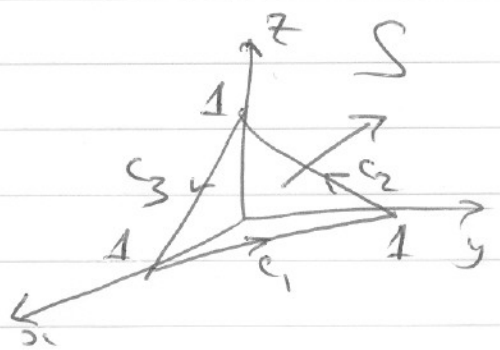
$$= 3 \int_{y=0}^{2} \left(e^{\frac{y-2}{2}} - e^0\right) dy = 3 \int_0^2 \left(e^{\frac{y-2}{2}} - 1\right) dy$$

which you can do.

#15

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$= -\vec{i} - \vec{j} - \vec{k}$$



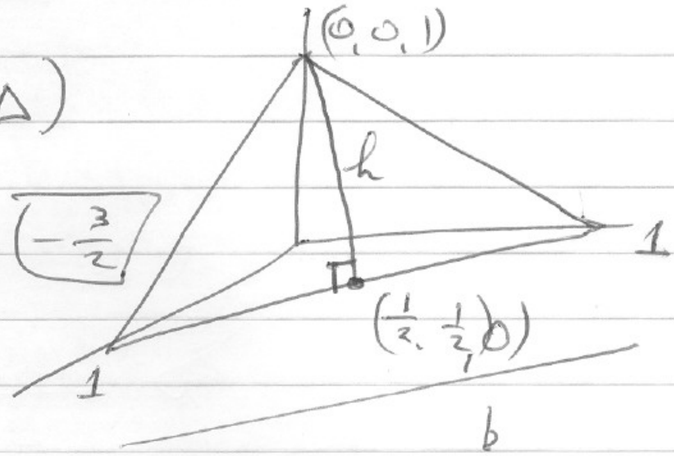

$\vec{n} = (1, 1, 1)/\sqrt{3}$

So
$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \iint_S (-1,-1,-1)\cdot\frac{(1,1,1)}{\sqrt{3}}\, dS$$

$$= \frac{-3}{\sqrt{3}}\iint_S dS = \frac{-3}{\sqrt{3}}\,\text{Area}(S)$$

$$= \frac{-3}{\sqrt{3}}\cdot\frac{1}{2}\, b\, h \qquad (S \text{ is a } \Delta)$$

$$= -\frac{1}{\sqrt{3}}\,\frac{3}{2}\,\sqrt{2}\,\sqrt{\frac{3}{2}} = -\frac{3\sqrt{3}}{2\sqrt{3}} = \boxed{-\frac{3}{2}}$$




as
$$h = \left|(0,0,1) - \left(\tfrac{1}{2},\tfrac{1}{2},0\right)\right| = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}+1} = \sqrt{\tfrac{3}{2}}$$

OR

$\boxed{C_1}$ $\quad \vec{r}(t) = (1-t,\ t,\ 0) \qquad 0\le t\le 1$

$\vec{r}\,'(t) = (-1,1,0)$

$$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_0^1 t(-1) + 0.1 + (1-t)0\, dt = \int_0^1 -t\,dt \Rightarrow$$

$$= \left[-t^2/2\right]_0^1 = -1/2$$

$\boxed{C_2}$

$\int_{C_2}\vec{F}\cdot d\vec{r} \qquad \vec{r}(t) = (0,\ 1-t,\ t) \quad 0\le t\le 1$

$$= \int_0^1 (1-t)\,0 + t(-1) + 0.1\, dt = -1/2 \qquad \vec{r}\,'(t) = (0,-1,1)$$

$\boxed{C_3}$ $\vec{r}(t) = (t, 0, 1-t)$ $\quad 0 \le t \le 1$

$\vec{r}\,'(t) = (1, 0, -1)$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^1 0.1 + (1-t)0 + t(-1)\, dt$$

$$= -1/2.$$

So $\boxed{\int_C \vec{F} \cdot d\vec{r} = -\frac{3}{2}}$ as $C = C_1 + C_2 + C_3$.

So $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = -\frac{3}{2} = \int_{\partial S} \vec{F} \cdot d\vec{r}$.

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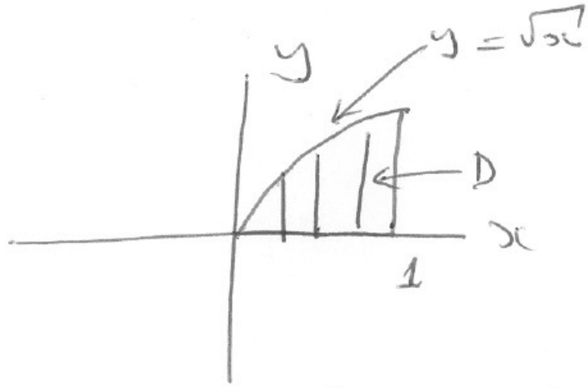
$\boxed{16.7}$

9




$\iiint_E 6xy\, dV$

$0 \le x \le 1$

$0 \le y \le \sqrt{x}$

$0 \le z \le 1+x+y$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{x}} \int_{z=0}^{1+x+y} 6xy\, dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{x}} 6xy(1+x+y)\, dy\, dx$$

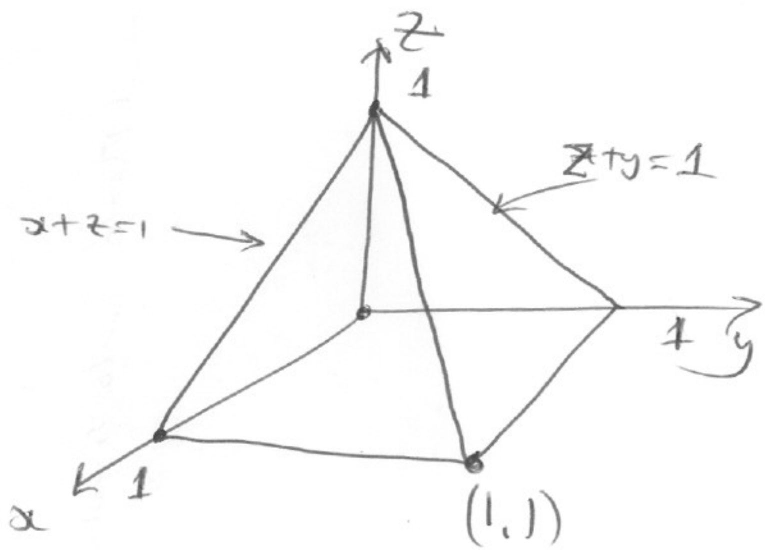
$$= \int_{x=0}^{1} \Big[ 3xy^2(1+x) + 2xy^3 \Big]_{y=0}^{y=\sqrt{x}} \, dx$$

$$= \int_0^1 3x^2(1+x) + 2x \cdot x^{3/2} \, dx$$

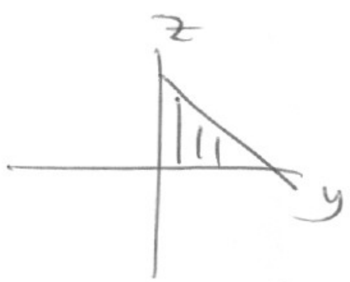
which you can do.

#13

E is pyramid




Fill it with matchsticks parallel to $x$ axis.




$0 \le y \le 1,$

$0 \le z \le 1-y$

$0 \le x \le 1-z$

Matchsticks start at $x=0$ and end at $x = 1-z$

$$\iiint_E z\,dV$$

$$= \int_{y=0}^{1} \int_{z=0}^{1-y} \int_{x=0}^{1-z} z\,dx\,dz\,dy$$

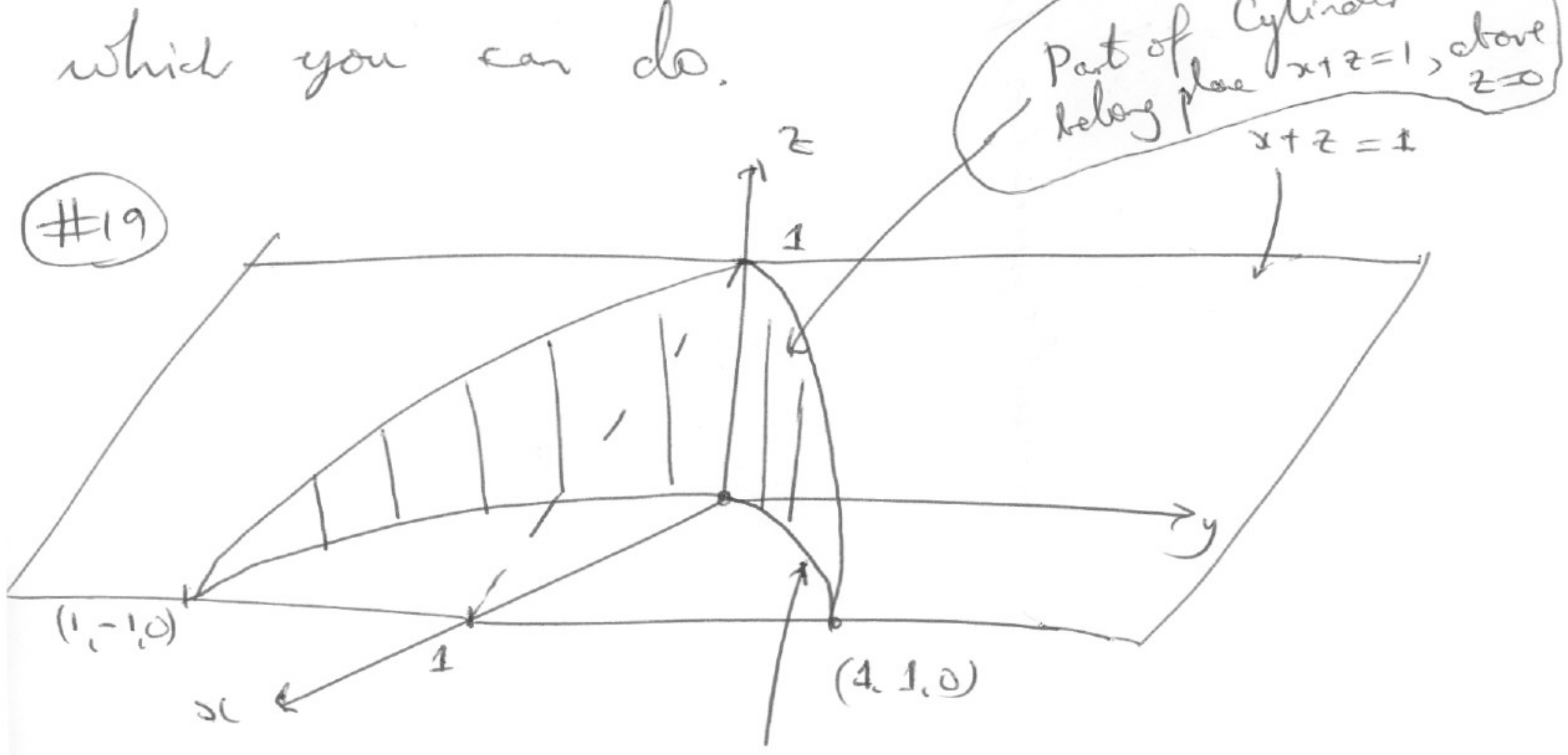
$$= \int_{y=0}^{1} \int_{z=0}^{1-y} z(1-z)\,dz\,dy$$

$$= \int_{y=0}^{1} \left[\frac{z^2}{2} - \frac{z^3}{3}\right]_0^{1-y} dy$$

$$= \int_0^1 \frac{(1-y)^2}{2} - \frac{(1-y)^3}{3}\,dy$$

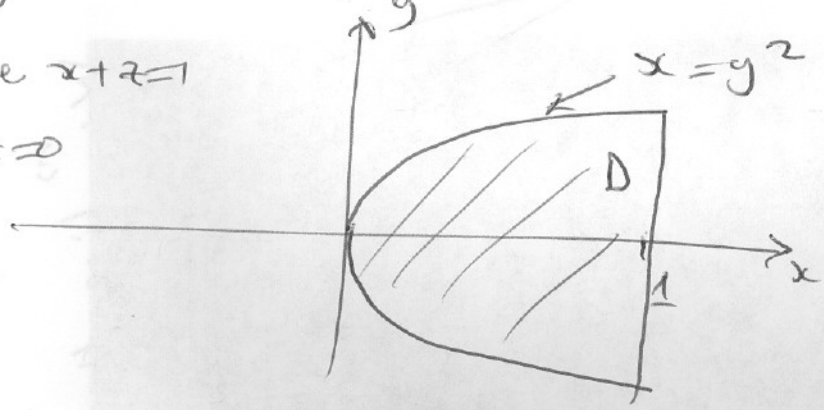
which you can do.

#19



So Region E is over Domain D (8)

under plane $x+z=1$
and over $z=0$




$$0 \le x \le 1$$

$$-\sqrt{x} \le y \le \sqrt{x}$$

$$0 \le z \le 1-x$$

$$\text{VOLUME}(E) = \int_{x=0}^{1} \int_{y=-\sqrt{x}}^{y=+\sqrt{x}} \int_{z=0}^{z=1-x} 1 \, dz \, dy \, dx$$

$$= \int_{x=0}^{x=1} \int_{y=-\sqrt{x}}^{y=\sqrt{x}} (1-x) \, dy \, dx$$

$$= \int_0^1 (1-x) \cdot 2\sqrt{x} \, dx$$

which you can do.