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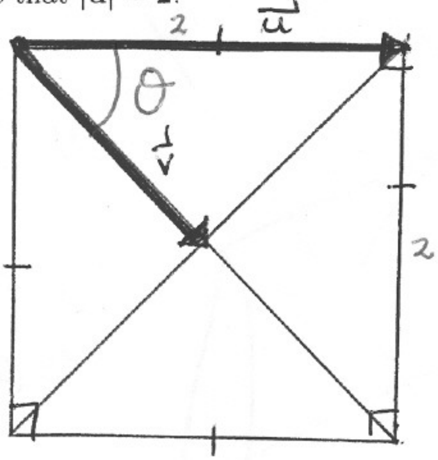
NAME: SOLUTIONS

1	/10
2	/15
3	/20
4	/10
5	/20
TOTAL	/75

MATH 251 (Spring 2004) Exam 1, Feb 25th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 65 minute exam. It is worth a total of 75 points.

(1) [10 pts] Let **u** and **v** be the vectors shown in the sketch and suppose that $|\mathbf{u}| = \mathbf{2}$.



$\theta = \pi/4$

$|\vec{v}| = \frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Use the geometric definition of the dot and cross products to find

(a) $\mathbf{u} \times \mathbf{v} = - |\vec{u} \times \vec{v}| \vec{k}$ by Right Hand Rule

$= - |\vec{u}||\vec{v}| \sin\theta \, \vec{k}$ 3+3

$= - 2 \cdot \sqrt{2} \sin \pi/4 \, \vec{k} = -2\vec{k}$

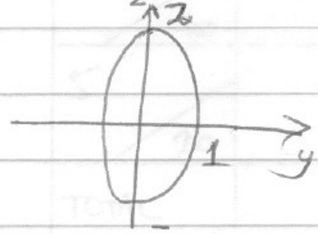
IF USE COORDS: Z OUT OF PAGE

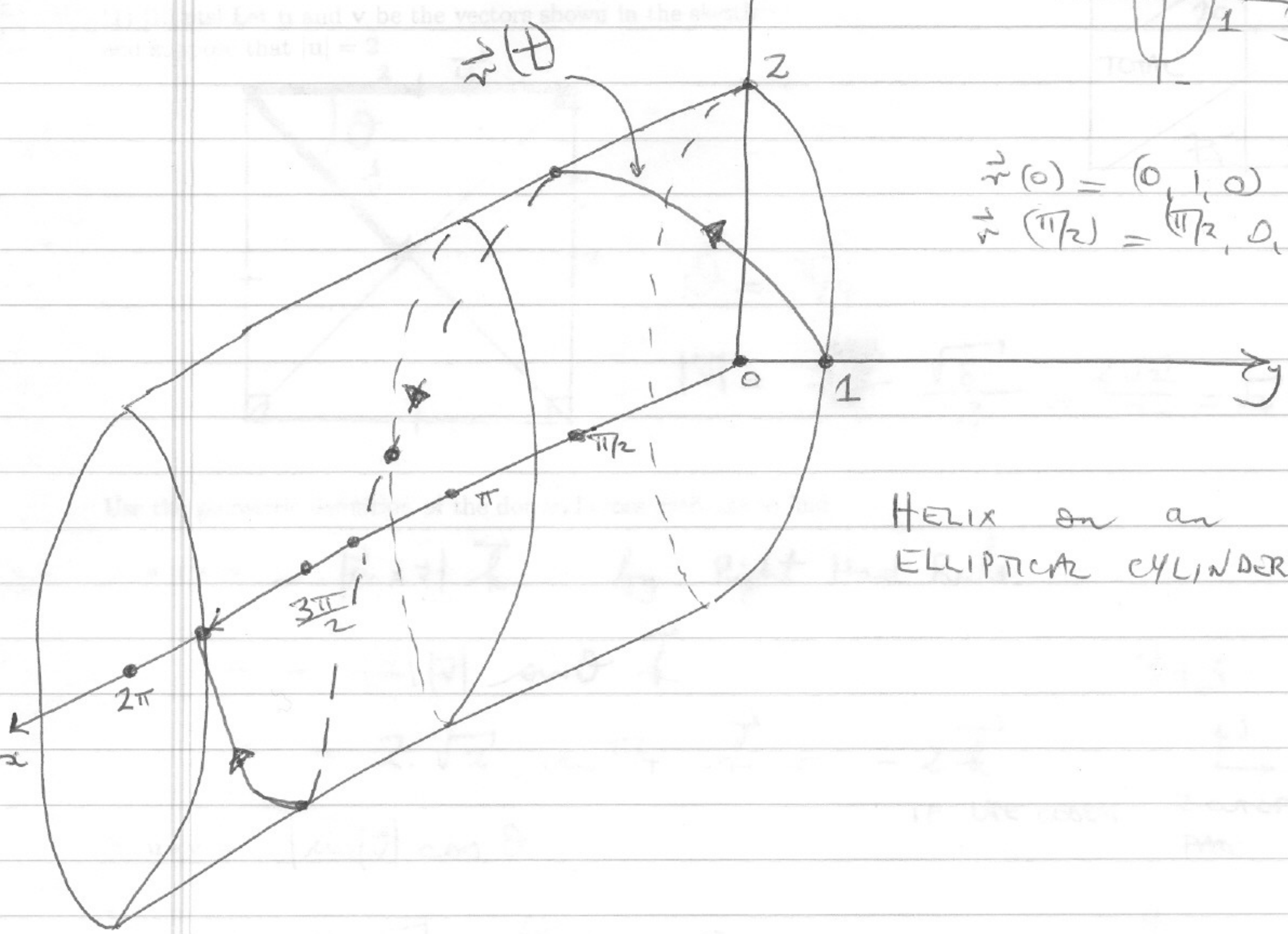
(b) $\mathbf{u} \bullet \mathbf{v} = |\vec{u}||\vec{v}| \cos\theta$

$= 2 \cdot \sqrt{2} \cos(\pi/4) = 2$ 4

2 (a) /7 $\quad x = t \qquad y = \cos t \qquad z = 2\sin t$

So $\quad \cos^2 t + \sin^2 t = 1 \;\Rightarrow\; y^2 + (z/2)^2 = 1$





$\vec{r}(0) = (0, 1, 0)$

$\vec{r}(\pi/2) = (\pi/2, 0, 2)$

HELIX on an ELLIPTICAL CYLINDER

(b) /3 $\quad \vec{r}\,'(t) = (1, -\sin t, 2\cos t)$

$\vec{r}\,'(0) = (1, 0, 2)$

(c) /5 $\quad \vec{l}(s) = \vec{r}(0) + s\,\vec{r}\,'(0) = (0, 1, 0) + s(1, 0, 2)$

$= (s, 1, 2s)$

(3) [20 pts]
Consider the plane through the points
$P = (1, 2, 3)$, $Q = (1, -2, -5)$, and $R = (3, 0, 7)$.

(a) Find a vector that is perpendicular to this plane.

$\vec{u} = \overrightarrow{PQ} = (1-1,\ -2-2,\ -5-3) = (0, -4, -8)$

and $\vec{v} = \overrightarrow{PR} = (3-1, 0-2, 7-3) = (2, -2, 4)$

lie in plane.

So normal vector is

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -4 & -8 \\ 2 & -2 & 4 \end{vmatrix} = -32\vec{i} - 16\vec{j} + 8\vec{k}$$

5

(b) Write down an equation of the form $ax + by + cz = d$ for this plane.

$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \qquad \vec{x} = (x, y, z) \qquad \vec{x}_0 = (1, 2, 3)$

$(-32, -16, 8) \cdot (x-1,\ y-2,\ z-3) = 0$

$-32(x-1) - 16(y-2) + 8(z-3) = 0$

$-32x - 16y + 8z + 40 = 0$

5

(c) Find a parametrization of this plane.

$\vec{r}(s,t) = \vec{r}_0 + s\vec{u} + t\vec{v} \qquad \vec{r}_0 = (1, 2, 3)$

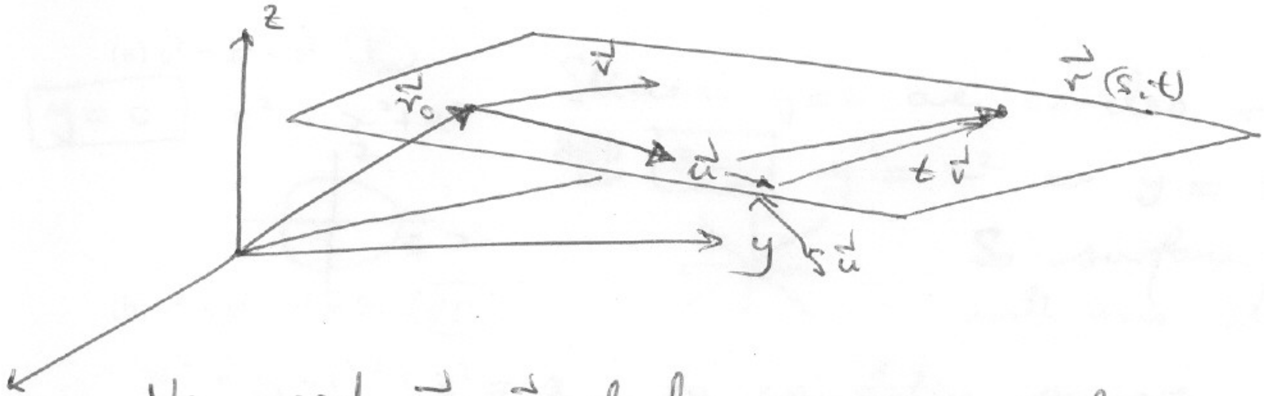
$\vec{u}, \vec{v}$ as in (a)

$\vec{r}(s,t) = (1, 2, 3) + s(0, -4, -8) + t(2, -2, 4)$

$= (1 + 2t,\ 2 - 4s - 2t,\ 3 - 8s + 4t)$

5

(d) Using a sketch and a couple of sentences, explain why $\mathbf{r}(s,t) = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$ is a parametrization of the plane through the endpoint of the vector $\mathbf{r}_0$ containing the vectors $\mathbf{u}$ and $\mathbf{v}$.



/5

We need $\vec{u}, \vec{v}$ to be non-zero vectors, and $\vec{u}, \vec{v}$ not to be parallel to each other. If this is the case then for any point in the plane there ~~are~~ $s, t \in \mathbb{R}$ so that $\vec{r}(s,t)$ is the ~~endpoint~~ position vector of the point, as we see using vector addition in sketch. Conversely, for any $s, t \in \mathbb{R}$, $\vec{r}(s,t)$ gives a point in the plane.

(4) [10 pts] Let P be the point with spherical coordinates $\rho = 4$, $\theta = \pi$, $\phi = \pi/6$. Find the rectangular and cylindrical coordinates of P.

RECTANGULAR

$$x = \rho \sin\phi \cos\theta = 4 \sin \pi/6 \cos \pi = +4 \cdot \tfrac{1}{2} \cdot (-1) = -2$$

$$y = \rho \sin\phi \sin\theta = 0 \quad \text{as } \sin \pi = 0$$

$$z = \rho \cos\phi = 4\cos(\pi/6) = 4 \cdot \sqrt{3}/2 = 2\sqrt{3}$$

$$\boxed{(x, y, z) = (-2, 0, 2\sqrt{3})}$$

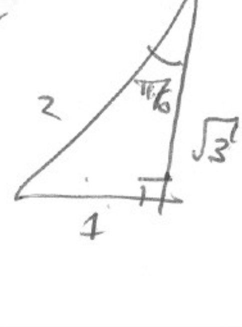
CYLINDRICAL

$$r = \rho \sin\phi = 4 \cdot \sin(\pi/6) = 4 \cdot \tfrac{1}{2} = 2$$

$$\theta = \pi$$

$$z = 2\sqrt{3}$$

$$\boxed{(r, \theta, z) = (2, \pi, 2\sqrt{3})}$$

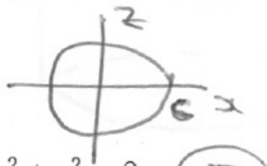


4 POINTS EACH

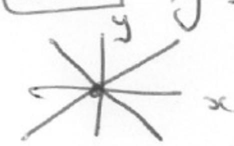
(5) [20 pts] Match the equations (a)-(e) with the graphs labeled (I)-(VI). [Note that there are more graphs than equations!] *Give reasons for your choices.*

(a) $y^2 = x^2 + z^2$ II

$\boxed{y = c}$ $c^2 = x^2 + z^2$ Slices in $y = c$ are circles radius c.



AND $\boxed{z = 0}$ $y^2 = x^2 \Rightarrow y = \pm x$



So surface is a double cone with axis along y axis

(b) $x^2 + y^2 + z^2 = 2y$ VI

$x^2 + (y-1)^2 + z^2 = 1$ by completing square.

Sphere center $(0,1,0)$ radius 1

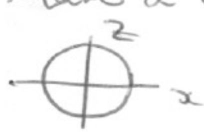
(c) $\rho = 1 \Rightarrow x^2 + y^2 + z^2 = 1$ Sphere center $(0,0,0)$ radius 1

III

(d) $y^2 = x^2 + z^2 + 1$ V

$\boxed{y = c}$ $x^2 + z^2 = c^2 - 1$

If $c^2 - 1 > 0$ have a circle radius $\sqrt{c^2-1}$ in xz plane



This occurs when $c > 1$ or $c < -1$

If $-1 < c < 1$ Then $c^2 - 1 < 0$ And $x^2 + z^2 = c^2 - 1$ is EMPTY SET

(e) $r = 2$ I

$r^2 = x^2 + y^2$

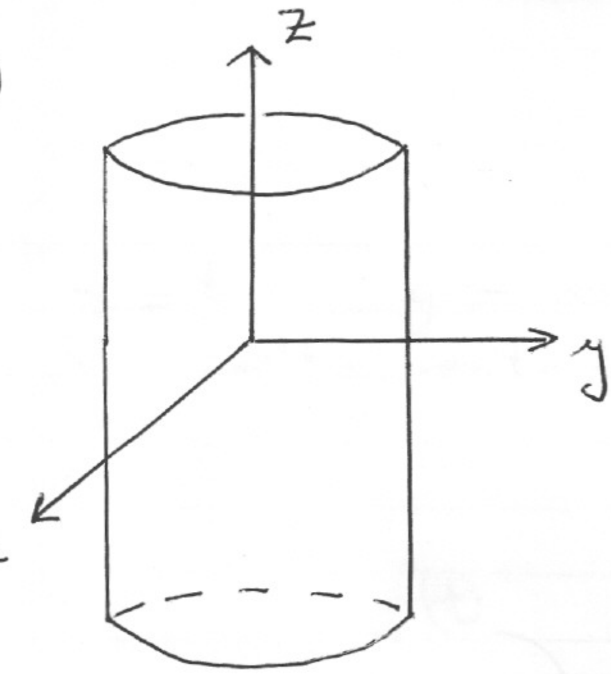
So $x^2 + y^2 = 4$

Cylinder ~~along~~ radius 2, axis ~~is~~ z axis

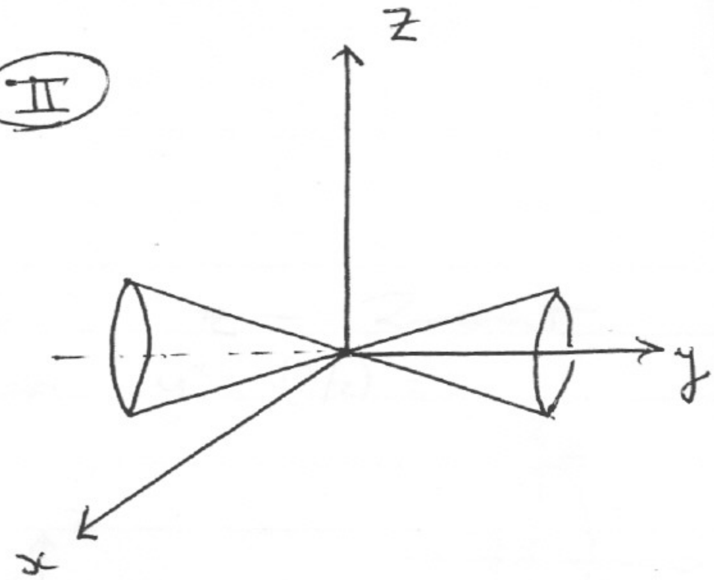
Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________

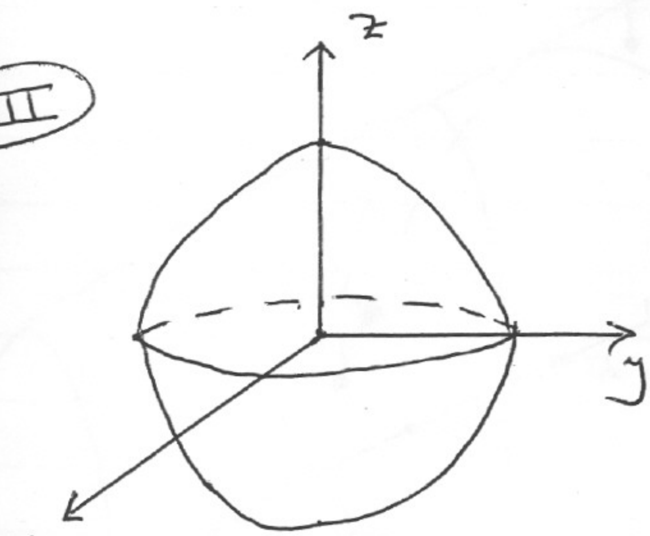
I



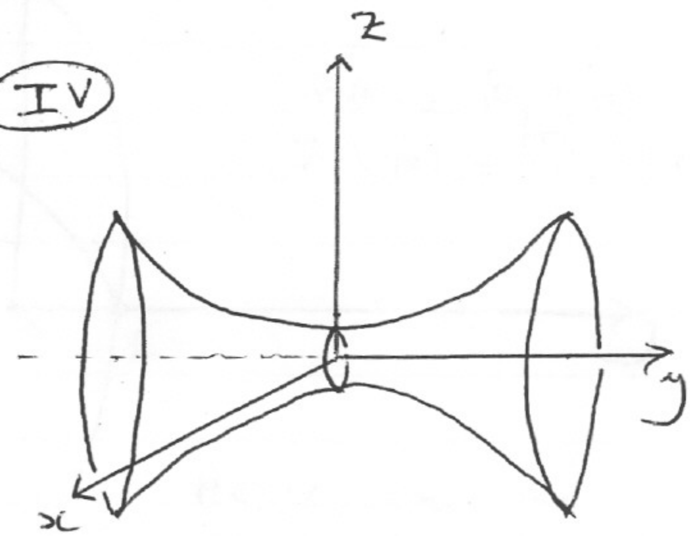
II



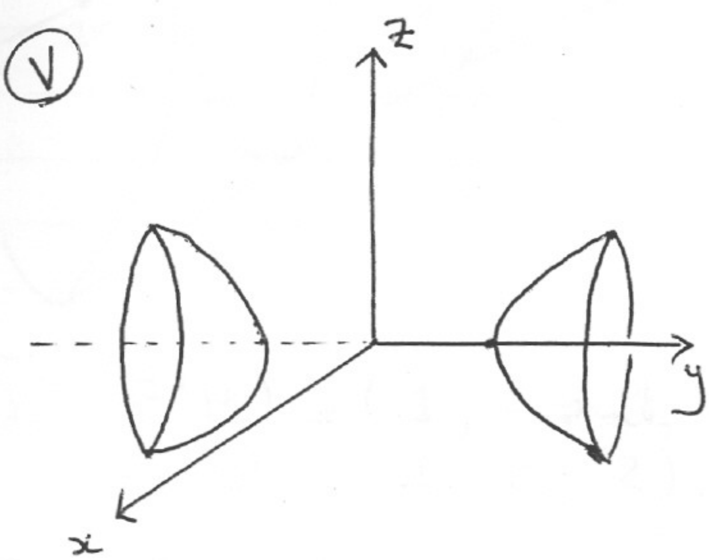
III



IV



V



VI