NAME: SOLUTIONS.

1	/8
2	/12
3	/12
4	/18
5	/15
6	/10
T	/75

MATH 251 (Spring 2004) Exam 2, March 31st

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 65 minute exam. It is worth a total of 75 points.

(1) [8 pts] Does the limit exist? Explain why, and if it does exist evaluate it.

$\lim_{(x,y)\to(0,0)} \frac{5xy}{x^2+3y^2}$

① Approaching (0,0) along line $y=0$:

$$\lim_{x\to 0} \frac{0}{x^2+0} = \lim_{x\to 0} 0 = 0$$

② Approaching (0,0) along line $y=x$:

$$\lim_{x\to 0} \frac{5x.x}{x^2+3x^2} = \lim_{x\to 0} \frac{5x^2}{4x^2} = \lim_{x\to 0} \frac{5}{4}$$

$$= 5/4$$

Since these two limits are not equal,

$$\lim_{(x,y)\to(0,0)} \frac{5xy}{x^2+y^2} \text{ does not exist.}$$

(2) [12 pts] Let $z = f(x,y) = 5x^2 + 4xy + 3y^2$ and let $\mathbf{r}(t) = (x(t), y(t))$ be a parametrization of a curve in the plane such that

$$\begin{aligned} \mathbf{r}(0) &= (-1,2), & \mathbf{r}(-4) &= (-6,8), \\ \mathbf{r}(7) &= (-1,3), & \mathbf{r}(4) &= (9,1), \\ \mathbf{r}'(0) &= (-4,7), & \mathbf{r}'(7) &= (-1,3), \\ \mathbf{r}'(-1) &= (4,5), & \mathbf{r}'(2) &= (-3,6). \end{aligned}$$

Let $g = f \circ \mathbf{r}$. Find $g'(0)$.

$$g'(0) = (f \circ \vec{r})'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$

NOW $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (10x + 4y,\ 4x + 6y)$$

$$\nabla f(\vec{r}(0)) = \nabla f(-1,2) = (10(-1) + 4 \cdot 2,\ 4 \cdot (-1) + 6 \cdot 2)$$
$$= (-2, 8)$$

And $$\vec{r}\,'(0) = (-4, 7)$$

So $$g'(0) = (-2, 8) \cdot (-4, 7) = 8 + 56 = \underline{64}$$

(3) [12 pts] Let $\mathbf{r}(t) = (\cos(\frac{3}{5}t), \sin(\frac{3}{5}t), \frac{4}{5}t)$ be a parametrization of a helix.

(a) Show that $\mathbf{r}$ is a unit speed curve

/5

$$\vec{r}\,'(t) = \left(-\frac{3}{5}\sin\frac{3}{5}t,\ \frac{3}{5}\cos\frac{3}{5}t,\ \frac{4}{5}\right)$$

$$\|\vec{r}\,'(t)\|^2 = \left(\frac{3}{5}\right)^2\left[\sin^2\left(\frac{3}{5}t\right) + \cos^2\left(\frac{3}{5}t\right)\right] + \left(\frac{4}{5}\right)^2$$

$$= \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$$

So $\|\vec{r}\,'(t)\| = 1, \Rightarrow \vec{r}$ is a unit speed curve

(b) Calculate the curvature of $\mathbf{r}$.

/7

$K(t) = \|\vec{r}\,''(t)\|$ since $\vec{r}$ is unit speed.

$$\vec{r}\,''(t) = \left(-\left(\frac{3}{5}\right)^2\cos\left(\frac{3}{5}t\right),\ -\left(\frac{3}{5}\right)^2\sin\left(\frac{3}{5}t\right),\ 0\right)$$

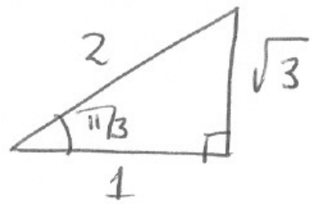
$$\|\vec{r}\,''(t)\|^2 = \left(\frac{3}{5}\right)^4\left[\cos^2\left(\frac{3}{5}t\right) + \sin^2\left(\frac{3}{5}t\right)\right] + 0$$

$$= \left(\frac{3}{5}\right)^4$$

So $\|\vec{r}\,''(t)\| = \left(\frac{3}{5}\right)^2 = \underline{\frac{9}{25}}$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$



(4) [18 pts] Let S the the surface which is the graph of the function $z = f(x, y) = x^2 + 4y^2$.

(a) Use the fact that $\mathbf{r}(u, v) = (u\cos v, \frac{1}{2}u\sin v, u^2)$ is a parametrization of S to find a parametrization of the tangent plane to S at the point where $(u, v) = (1, \frac{\pi}{3})$.

Tangent Plane has parametrization

$$P(s,t) = \vec{r}(1, \pi/3) + s\frac{\partial\vec{r}}{\partial u}(1, \pi/3) + t\frac{\partial\vec{r}}{\partial v}(1, \pi/3)$$

$$\vec{r}(1, \pi/3) = (1\cos\pi/3,\ \tfrac{1}{2}1\sin\pi/3,\ 1^2)$$
$$= \left(\frac{1}{2},\ \frac{\sqrt{3}}{4},\ 1\right)$$

$$\frac{\partial\vec{r}}{\partial u} = \left(\cos v,\ \tfrac{1}{2}\sin v,\ 2u\right)$$

$$\frac{\partial\vec{r}}{\partial u}(1, \pi/3) = \left(\cos\pi/3,\ \tfrac{1}{2}\sin\pi/3,\ 2.1\right) = \left(\frac{1}{2},\ \frac{\sqrt{3}}{4},\ 2\right)$$

$$\frac{\partial\vec{r}}{\partial v} = \left(-u\sin v,\ \tfrac{1}{2}u\cos v,\ 0\right)$$

$$\frac{\partial\vec{r}}{\partial v}(1, \pi/3) = \left(-\frac{\sqrt{3}}{2},\ \frac{1}{4},\ 0\right)$$

So

$$P(s,t) = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}, 1\right) + s\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, 2\right) + t\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}, 0\right)$$

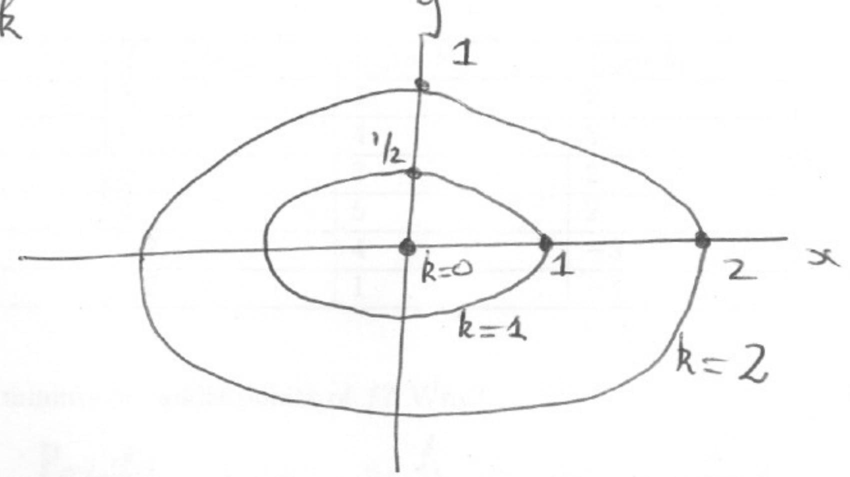
(b) Sketch the level curves of f at levels $z = 0, 1, 4$.

$$f(x,y) = x^2 + 4y^2 = k$$

$\boxed{=1}$

$$x^2 + 4y^2 = 1$$

$$x^2 + \left(\frac{y}{\frac{1}{2}}\right)^2 = 1$$



$\boxed{=4}$

$$x^2 + 4y^2 = 4$$

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

(c) In what direction in the xy-plane is the rate of change of f minimized at $(x, y) = (1, 1)$. What is the value of this minimum rate of change?

Direction is $\vec{v} = -\dfrac{\nabla f(1,1)}{|\nabla f(1,1)|}$

NOW

$$\nabla f(x,y) = (2x, 8y)$$

$$\nabla f(1,1) = (2, 8)$$

$$|\nabla f(1,1)| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

So $\vec{v} = -\dfrac{1}{\sqrt{20}}(2,8) = -\dfrac{1}{\sqrt{5}}(1,4)$

VALUE OF MIN RATE OF CHANGE IS $-|\nabla f(1,1)| = -2\sqrt{5}$

(5) [15 pts] Suppose a function $z = f(x, y)$ has continuous second partial derivatives and that

(a,b)	$f(a,b)$	$\nabla f(a,b)$	$f_{xx}(a,b)$	$f_{xy}(a,b)$	$f_{yy}(a,b)$
$(1,2)$	0	$(0,0)$	8	4	2
$(3,4)$	0	$(1,4)$	6	4	5
$(5,6)$	3	$(0,0)$	5	3	2
$(7,8)$	-5	$(2,3)$	1	5	2
$(9,10)$	1	$(0,0)$	-2	4	-3
$(11,12)$	2	$(0,0)$	-2	1	-3

Which of the points (a, b) are local maxima, minima or saddle points of f? Why?

Local Max, Min + Saddle Points can only occur ~~where~~ at critical points of f, i.e. (a,b) where $\nabla f(a,b) = (0,0)$.

So candidates are $(a,b) = (1,2)$, $(5,6)$, $(9,10)$, $(11,12)$.

APPLY 2ND DERIVATIVE TEST

$\boxed{(1,2)}$ $D = \det\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \det\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} = 0$

NO CONCLUSION POSSIBLE

$\boxed{(5,6)}$ $D = \det\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = 10 - 9 = 1 > 0.$

$f_{xx} = 5 > 0$ So LOCAL MIN at $(5,6)$

$\boxed{(9,10)}$ $D = \det\begin{bmatrix} -2 & 4 \\ 4 & -3 \end{bmatrix} = 6 - 16 = -10 < 0$

Saddle Point at $(9,10)$

$\boxed{(11,12)}$ $D = \det\begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} = 6 - 1 = 5 > 0$ $f_{xx} = -2 < 0$

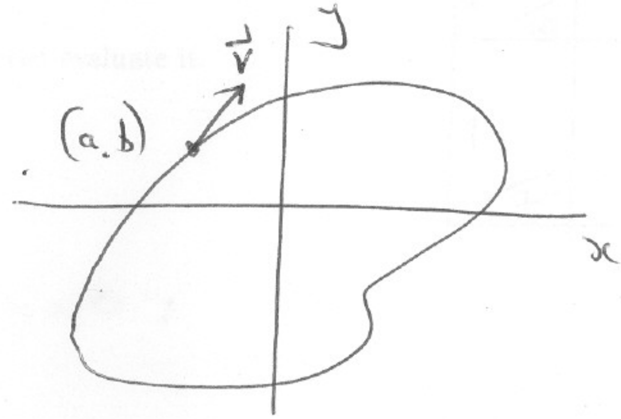
LOCAL MAX AT $(11,12)$

(6) [10 pts] Let $z = f(x, y)$. Prove that the gradient vector $\nabla f(a, b)$ is perpendicular to the level curve of f through the point (a, b).

Hint: Let $\mathbf{r}(t)$ be a parametrization of the level curve to f through (a, b). What do you know about $f(\mathbf{r}(t))$?

Suppose $\vec{r}(0) = (a,b)$

and $\vec{v} = \vec{r}\,'(0)$ is tangent to level curve.



We must show $\nabla f(a,b)$ is perpendicular to $\vec{v}$.

Well $f(\vec{r}(t)) = k$ is a constant as $\vec{r}$ is a level curve of f.

So
$$0 = (f \circ \vec{r})'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \quad \text{by CHAIN RULE}$$

Put $t=0$

$$\nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0) = 0$$
$$\nabla f(a,b) \cdot \vec{v} = 0$$

So $\nabla f(a,b)$ is perpendicular to $\vec{v}$.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________