MATH 251 (Spring 2004) Exam 3, April 28th

No calculators, books or notes!
Show all work and give **complete explanations** for all your answers.
This is a 65 minute exam. It is worth a total of 75 points.

(1) [14 pts] Set up integrals of the form

$$\int_{t=a}^{t=b} h(t)\,dt$$

that are equal to the following integrals, but do NOT evaluate the integrals you set up.

(a) $\int_C \ln(x+y)ds$, where C is the curve which is arc of the parabola $y = x^2$ from $(1,1)$ to $(3,9)$. 7

$\vec{r}(t) = (t, t^2) \quad 1 \leq t \leq 3.$

$\vec{r}\,'(t) = (1, 2t) \quad \|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$

$\int_C \ln(x+y)\,ds = \int_{t=1}^{t=3} \ln(t+t^2)\sqrt{1+4t^2}\,dt$

(b) $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve parametrized by $r(t) = (1+2t, 3+4t^2)$ and $\mathbf{F}(x,y) = x^2\mathbf{i} + \sin(y)\mathbf{j}$. 7

$0 < t < 2$

$\vec{r}\,'(t) = (2, 8t)$

$\vec{F}(\vec{r}(t)) = (1+2t)^2\vec{\imath} + \sin(3+4t^2)\vec{\jmath}$

$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{t=2} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

$= \int_{t=0}^{2} [2(1+2t)^2 + 8t\sin(3+4t^2)]\,dt$

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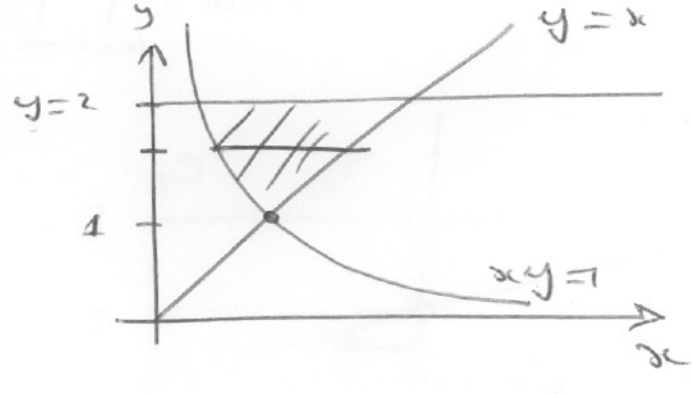
(2) [13 pts]
(a) Calculate $\iint_D y\,dA$, where D is the region in the first quadrant of the xy-plane that lies above the hyperbola $xy = 1$, above the line $y = x$ and below the line $y = 2$.

TYPE II REGION
STRAIGHT TOP + BOTTOM



$1 \le y \le 2$

$\frac{1}{y} \le x \le y$

$$\iint_D y\,dA = \int_{y=1}^{y=2} \int_{x=\frac{1}{y}}^{x=y} y\,dx\,dy$$

$$= \int_{y=1}^{y=2} y\,[x]_{x=\frac{1}{y}}^{x=y}\,dy$$

$$= \int_{y=1}^{y=2} y\left[y - \frac{1}{y}\right] dy$$

$$= \int_1^2 (y^2 - 1)\,dy$$

$$= \left[\frac{y^3}{3} - y\right]_1^2 = \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)$$

$$= 7/3 - 1 = \boxed{4/3}$$

5

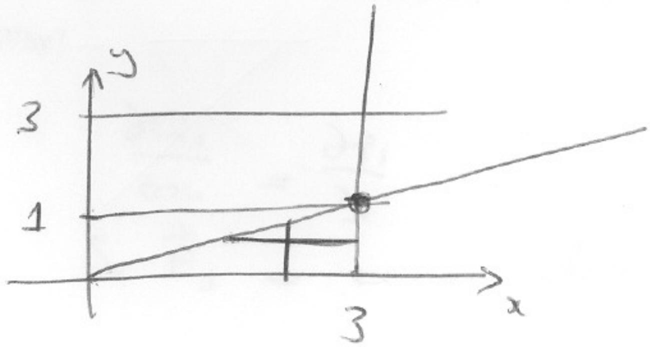
(b) Find a, b, $f_1(x)$ and $f_2(x)$ so that

$$y=1 \quad \int_{y=0}^{y=3}\int_{x=3y}^{x=3} e^{x^2}\,dx\,dy = \int_{x=a}^{x=b}\int_{y=f_1(x)}^{y=f_2(x)} e^{x^2}\,dy\,dx$$

STRAIGHT TOP + BOTTOM

$$0 \le y \le 1$$

$$3y \le x \le 3$$



STRAIGHT SIDES

$$y = \frac{x}{3}$$

$$0 \le x \le 3$$

$$0 \le y \le \frac{x}{3}$$

$$\int_{x=0}^{x=3}\int_{y=0}^{y=\frac{x}{3}} e^{x^2}\,dy\,dx$$

$a=0$ $\quad f_1(x)=0$

$b=3$ $\quad f_2(x)=\frac{x}{3}$

(3) [14 pts] Consider the two vector fields

$$\begin{aligned}
\mathbf{F}_1(x,y) &= (2xy - 2y^2\sin x)\mathbf{i} \;+\; (x^2 + 4y\cos x)\mathbf{j} \\
\mathbf{F}_2(x,y) &= (2xy^2 - 2y\sin x)\mathbf{i} \;+\; (x^2 + 4y^2\cos x)\mathbf{j}
\end{aligned}$$

One of these vector fields is conservative.

(a) Which vector field is conservative and which is not? Why?

$\frac{\partial Q_1}{\partial x} = 2x + -4y \sin x$ $\qquad \frac{\partial Q_1}{\partial x} = \frac{\partial P_1}{\partial y}$

$\frac{\partial P_1}{\partial y} = 2x - 4y \sin x$ $\qquad$ So $\vec{F}_1$ is conservative

$\frac{\partial Q_2}{\partial x} = 2x - 4y^2 \sin x$ $\qquad \frac{\partial Q_2}{\partial x} \neq \frac{\partial P_2}{\partial y}$ $\qquad$ 6

$\frac{\partial P_2}{\partial y} = 4xy - 2\sin x$ $\qquad$ So $\vec{F}_2$ is NOT conservative

(b) For the vector field that is conservative, evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any curve from $(0,0)$ to $(0,1)$.

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FIND f:

$\vec{F} = \nabla f$

$\frac{\partial f}{\partial x} = P = 2xy - 2y^2 \sin x \Rightarrow f(x,y) = x^2y + 2y^2\cos x + g_1(y)$

$\frac{\partial f}{\partial y} = Q = x^2 + 4y\cos x \Rightarrow f(x,y) = x^2y + 2y^2\cos x + g_2(x)$

So $\boxed{f(x,y) = x^2y + 2y^2\cos x}$

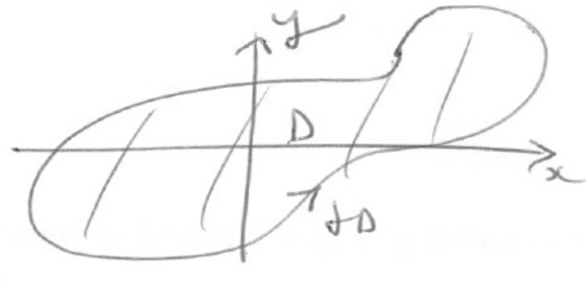
By FTC for Line Integrals

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(0,1) - f(0,0)$

$= 2 - 0 = \boxed{2}$

(4) [12 pts]

(a) Carefully state Green's Theorem



Let D be an open set in $\mathbb{R}^2$ with boundary curve ∂D. Orient ∂D so that as you walk around ∂D with head ~~facing~~ in $+z$ direction the region D is on your left, ie ∂D is positively oriented. Let $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ be a vector field on D so that P, Q have continuous partial derivatives

The $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$ 6

(b) Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2dy$, where C is the circle $x^2 + y^2 = 4$ with counter-clockwise orientation.

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$$\int_C \overset{P}{x^2y}\,dx - \overset{Q}{xy^2}\,dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D -y^2 - x^2 \, dA$$

$$= -\iint_D x^2+y^2 \, dx\,dy = -\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^2 . \, r\,dr\,d\theta$$

$$= -2\pi \int_{r=0}^{2} r^3\,dr$$

$$= -2\pi \left[\frac{r^4}{4}\right]_0^2 = -2\pi . 4 = \boxed{-8\pi}$$

(5) [12 pts] Use the Method of Largange Multipliers to maximize the function $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 16$. [Hint: There are 4 critical points.]

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 16 \end{cases}$$

$g = 4x^2 + y^2$

$y = 8x\lambda$ ①

$x = 2y\lambda$ ②

$4x^2 + y^2 = 16$ ③

By ① $yx = 8x^2\lambda$

By ② $yx = 2y^2\lambda$

$0 = 2\lambda(4x^2 - y^2)$

So $\lambda = 0$ or $y^2 = 4x^2$.

$\boxed{\lambda = 0}$ $y = 0$, $x = 0$ by ①, ②. But then ③ doesn't hold. NO SOLUTIONS

$\boxed{y^2 = 4x^2}$ plug into ③ $8x^2 = 16$ $x = \pm\sqrt{2}$

$x = \pm\sqrt{2}$, $y = \pm 2\sqrt{2}$. $\lambda = \frac{x}{2y} = \frac{\pm\sqrt{2}}{\pm 2 \cdot 2\sqrt{2}} = \pm\frac{1}{4}$

CRITICAL POINTS (x, y, λ)

x	y	λ	$f(x,y)$
$\sqrt{2}$	$2\sqrt{2}$	$\frac{1}{4}$	4
$\sqrt{2}$	$-2\sqrt{2}$	$-\frac{1}{4}$	-4
$-\sqrt{2}$	$2\sqrt{2}$	$-\frac{1}{4}$	-4
$-\sqrt{2}$	$-2\sqrt{2}$	$\frac{1}{4}$	4

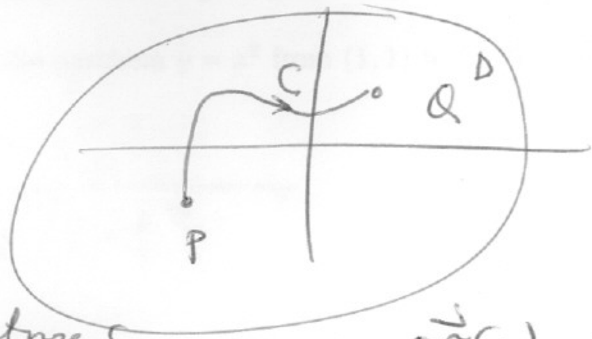
SO MAX IS AT $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$ and f has value 4 there.

(6) [10 pts] State and prove the Fundamental Theorem of Calculus for Line Integrals.

Let f be a differentiable function on an open set D in $\mathbb{R}^2$, and let C be an oriented curve from P to Q which is contained in D

Then

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$$



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PF Let $\vec{r}(t)$, $a \le t \le b$ parametrize C. $\vec{r}(a) = P$, $\vec{r}(b) = Q$

$$\int_C \nabla f \cdot d\vec{r} = \int_{t=a}^{t=b} \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\overset{\text{CHAIN RULE}}{=} \int_{t=a}^{t=b} (f \circ \vec{r})'(t)\, dt$$

$$\underset{\text{IN 1 VARIABLE}}{\overset{\text{FTC}}{=}} (f \circ \vec{r})(b) - (f \circ \vec{r})(a)$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(Q) - f(P)$$

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Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________