MATH 251H (Fall 2003) Final Exam, Dec 15th

No calculators, books or notes! Show all your work.
This 120 minute exam is worth 150 points (10 points = 8 minutes).

(1) [8 pts] Let $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ be two vectors.

(a) Calculate the (vector) projection of $\mathbf{u}$ onto $\mathbf{v}$.

(b) Find a vector that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

(c) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

(2) [8 pts] Sketch the following surfaces

(a) $\theta = \pi/3$

(b) $\phi = \pi/3$

(c) $\rho = \pi/3$

(d) $\rho \sin \phi = \pi/3$

(3) [12 pts] Let C be the curve with parametrization $\mathbf{r}(t) = t \sin t\, \mathbf{i} + t \cos t\, \mathbf{j}$ for $0 < t < 4\pi$.

(a) Sketch the curve C, indicating the direction of increasing t.

(b) Find a parametrization of the tangent line to the curve C at $t = \frac{\pi}{4}$.

(c) Calculate $\int_C \sqrt{x^2 + y^2}\, ds$

(4) [12 pts] Consider the parametrized surface $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u\mathbf{k}$, where $0 < u \leq 2$, and $0 \leq v < 2\pi$.

(a) Find a parametrization of the tangent plane to the surface at $u = 1$, $v = \frac{\pi}{2}$.

(b) Find an equation of the form $f(x, y, z) = 0$ for this surface.

(c) Find an equation of the form $ax + by + cz + d = 0$ of the tangent plane in (a).

(d) Sketch the surface and the tangent plane.

(5) [8 pts] Suppose that $z = f(x, y)$ and $(x, y) = \mathbf{r}(u, v)$. Let $g = f \circ \mathbf{r}$. Find $\frac{\partial g}{\partial u}(1, 3)$ if

$$\mathbf{r}(1,3) = (2,4) \qquad \frac{\partial \mathbf{r}}{\partial u}(1,3) = (5,-3) \qquad \frac{\partial \mathbf{r}}{\partial v}(1,3) = (6,7)$$

$$\mathbf{r}(2,4) = (1,3) \qquad \frac{\partial \mathbf{r}}{\partial u}(2,4) = (4,5) \qquad \frac{\partial \mathbf{r}}{\partial v}(2,4) = (9,7)$$

$$\frac{\partial f}{\partial x}(1,3) = -4 \qquad \frac{\partial f}{\partial y}(1,3) = 7$$

$$\frac{\partial f}{\partial x}(2,4) = 8 \qquad \frac{\partial f}{\partial y}(2,4) = 6$$

(6) [12 pts] Suppose that $z = f(x, y)$ is a function whose second partial derivatives are continuous at all points in the plane and that

$$\begin{array}{llll}
\bigtriangledown f(2,4) = (0,0) & f_{xx} = 8 & f_{xy} = 4 & f_{yy} = 2 \\
\bigtriangledown f(3,5) = (0,0) & f_{xx} = 8 & f_{xy} = 3 & f_{yy} = 2 \\
\bigtriangledown f(4,6) = (0,0) & f_{xx} = 8 & f_{xy} = 5 & f_{yy} = 2 \\
\bigtriangledown f(5,7) = (0,0) & f_{xx} = -8 & f_{xy} = 3 & f_{yy} = -2 \\
\bigtriangledown f(6,8) = (0,2) & f_{xx} = -8 & f_{xy} = 5 & f_{yy} = -2.
\end{array}$$

Identify any local maxima, minima, and saddle points of f. Explain the reasons for your answers.

(7) [15 pts] The base of an aquarium with a given volume V is made of slate and the sides are made of glass. (There is no top.) If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimze the cost of the materials.

(8) [10 pts] Show that $\int\limits_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(9) [15 pts] Set up an iterated integral for $\iiint_E z\,dV$, where E is the solid region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $y + z = 1$, and $x + z = 1$. *Do* not *evaluate the integral.*

(10) [16 pts] (a) State Green's Theorem

(b) Verify Green's Theorem for the vector field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$ and the domain D bounded by the paraboloid $y = x^2$ from $(0,0)$ to $(2,4)$, and the line segments from $(2,4)$ to $(0,4)$, and from $(0,4)$ to $(0,0)$.

(11) [15 pts] Let $\mathbf{F}$ be the vector field $\mathbf{F} = 2yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$.
(a) Calculate $\bigtriangledown \bullet \mathbf{F}$

(b) Calculate $\bigtriangledown \times \mathbf{F}$

(c) Determine whether or not $\mathbf{F}$ is conservative. If it is conservative, find a function f so that $\mathbf{F} = \bigtriangledown f$.

(d) Is there a vector field $\mathbf{G}$ so that $\mathbf{F} = \bigtriangledown \times \mathbf{G}$? Why?

(12) [6 pts] Evaluate $\iint_D x^2 \tan x + y^3 + 4dA$, where $D = \{(x, y)|x^2 + y^2 \leq 2\}$.

(13) [13 pts]

(a) Sketch an example of a vector field $\mathbf{F}$ and a curve C so that
(i) $\int_C \mathbf{F} \bullet d\mathbf{r} < 0$

(ii) $\int_C \mathbf{F} \bullet d\mathbf{r} = 0$

(b) *State and prove* the Fundamental Theorem of Calculus for line integrals of vector fields.

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________