NAME: SOLUTIONS

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MATH 251 (Spring 2008) Exam 3, Apr

No calculators, books or notes! Show all work and give **complete explan**
This 65 minute exam is worth a total of 75 points.

(1) [12 pts] Let C be the curve in the plane which is parametrized by $\mathbf{r}(t)$
and let $z = f(x, y) = xy$. Calculate $\int_C f\,ds$.

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

Now $\vec{r}\,'(t) = (3, 4)$ so $\|\vec{r}\,'(t)\| =$

and $f(\vec{r}(t)) = (3t)(4t+1) = 12t^2 + 3t$

So

$$\int_C f\,ds = \int_0^1 (12t^2 + 3t)\,5\,dt$$

$$= 60\int_0^1 t^2\,dt + 15\int_0^1 t\,d$$

$$= 60\left[t^3/3\right]_0^1 + 15\left[t^2/\right.$$

$$= 60 \cdot \tfrac{1}{3} + 15 \cdot \tfrac{1}{2} =$$

$=$

(2) [16 pts]

(a) Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y) = (2x + e^x \cos y)\mathbf{i} + (3y^2 - e^x \sin y)\mathbf{j}. = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

Show that $\mathbf{F}$ is conservative on the domain $D = \mathbf{R}^2$.

The domain $D = \mathbb{R}^2$ is open and simply connected. So by a theorem from class we just need to show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, since then $\vec{F}$ will be conservative.

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3y^2 - e^x \sin y) = -e^x \cos y$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2x + e^x \cos y) = -e^x \cos y$$

So $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ does hold, and therefore $\vec{F}$ is conservative

(b) Let C be the curve in the plane which is parametrized by $\mathbf{r}(t) = (\cos t, \sin t)$, where $0 \le t \le \pi$ and let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y) = y\mathbf{i} + e^x\mathbf{j}$. Find a formula for a function $g(t)$ so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} g(t)\, dt.$$

~~[Do NOT attempt to evaluate the integral on the right-hand side of the equation above.]~~

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^{\pi} g(t)\, dt$$

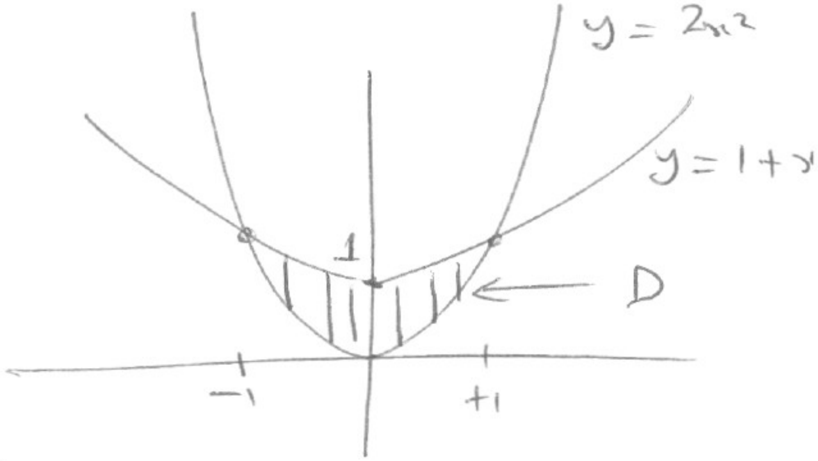
So $g(t) = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$

Now $\vec{r}\,'(t) = (-\sin t, \cos t)$

$\vec{F}(\vec{r}(t)) = \vec{F}(\cos t, \sin t) = (\sin t, e^{\cos t})$

So $g(t)$ [illegible] $\cos t$

(3) [16 pts] (a) Let D be the region in the plane bounded by the curves $y = 1 + x^2$, $y = 2x^2$ ~~and the y-ax~~
Calculate $\iint_D x^2 \, dA$.



The curves meet whee

$1 + x^2 = 2x^2$

$x^2 = 1$

$x = \pm 1.$

The region D is a Type I region.

Since $1 + x^2 > 2x^2$ at $x = 0$ $(1 = 1 + 0^2 > 2 \cdot 0^2 = 0$

we know $y = 1 + x^2$ is top curve and $y = 2x^2$

So D is given by

$-1 \leq x \leq 1$

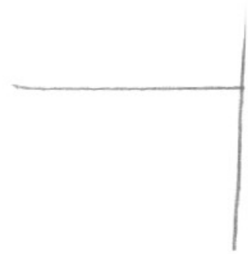
$2x^2 \leq y \leq 1 + x^2$

Then $\iint_D x \, dA = \int_{x=1}^{x=+1} \int_{y=2x^2}^{y=1+x^2} x^2 \, dy \, dx$

$= \int^1 x^2 (1 + x^2 - 2x^2) \, dx = \int_{-1}^{1} (x^2 - x^4)$

(b) Let D be the region in the first quadrant (i.e., $x \geq 0$ and $y \geq 0$) that is b
and $x^2 + y^2 = 4$. Calculate $\iint_D y\, dA$.

This region is nicely described in polar coordinates as



$0 \leq \theta \leq \pi/2$

$1 \leq r \leq 2$ as x^2

So $\iint_D y\, dA = \int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=2} r \sin($

$= \left(\int_0^{\pi/2} \sin\theta \, d\theta \right) \left(\int_1 \right.$

$= \left[-\cos\theta \right]_0^{\pi/2} \left[\frac{r^3}{3} \right]$

$= (0+1) \cdot \left(\frac{8}{3} - \frac{1}{3} \right) =$

(4) [16 pts] Let S be the surface in $\mathbf{R}^3$ that is parametrized by

$$\mathbf{r}(u, v) = (v \cos u, v \sin u, v), := (x, y, z)$$

where $0 \le u \le 2\pi$ and $v > 0$.

(a) Show that S is the cone $z = \sqrt{x^2 + y^2}$.

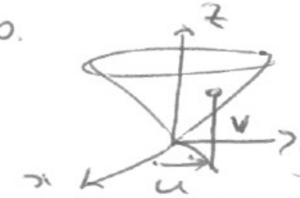
$x = v\cos u$
$y = v \sin u$
$z = v$

So $x^2 + y^2 = v^2 \cos^2 u + v^2 \sin^2 u = v^2$

$\sqrt{x^2+y^2} = \sqrt{v^2} = v$ as $v > 0$

So $z = \sqrt{x^2+y^2}$ holds.

Since $0 < u < 2\pi$ and $v > 0$ we get the entire cone.



(b) Use the parametrization $\mathbf{r}(u, v)$ above to calculate a *parametrization* of the tangent plane to S at the point $\mathbf{r}(\pi/4, 1)$.

$\vec{r}(\pi/4, 1) = (1 \cos \pi/4, 1 \sin \pi/4, 1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$

$\frac{\partial \vec{r}}{\partial u} = (-v \sin u, v \cos u, 0)$, $\frac{\partial \vec{r}}{\partial u}(\pi/4, 1) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$\frac{\partial \vec{r}}{\partial v} = (\cos u, \sin u, 1)$ $\frac{\partial \vec{r}}{\partial v}(\pi/4, 1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$

So parametrization of tangent plane is

$P(s,t) = \vec{r}(\pi/4, 1) + s \frac{\partial \vec{r}}{\partial u}(\pi/4, 1) + t \frac{\partial \vec{r}}{\partial v}(\pi/4, 1)$

$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) + s\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$

$= \frac{1}{\sqrt{2}}(1 - s + t)\vec{i} + \frac{1}{\sqrt{2}}(1 + s + t)\vec{j} + (1 + t)\vec{k}$

(5) [15 pts] Use the Method of Lagrange Multipliers to find the maximum and minimum values of the function $f(x, y) = 6x + 8y$ subject to the constraint $x^2 + y^2 = 1$.

$\left\{ \begin{array}{l} g(x,y) = x^2+y^2 = 1 \\ \nabla f = \lambda \nabla g \end{array} \right\}$ gives $\begin{array}{l} x^2+y^2=1 \\ 6 = \lambda 2x \\ 8 = \lambda 2y \end{array}$

or
$3 = \lambda x$ ①
$4 = \lambda y$ ②
$x^2+y^2 = 1$ ③

~~The~~ $4 \times$ ① $- 3 \times$ ② gives

$4\times 3 - 3\times 4 = 4\lambda x - 3\lambda y$
$0 = \lambda(4x-3y)$

So $\lambda = 0$ or $4x = 3y$

$\boxed{\lambda=0}$ Impossible as ~~the~~ ①, ② are not satisfied.

$\boxed{4x=3y}$ $y = \frac{4}{3}x$. plug into ③ to get

$x^2 + \left(\frac{4}{3}\right)^2 x^2 = 1 \Rightarrow x^2 = \frac{1}{1+\left(\frac{4}{3}\right)^2} = \frac{3^2}{3^2+4^2} = \frac{3^2}{5^2}$

So $\boxed{x = \pm\frac{3}{5}}$ and $y = \pm\frac{4}{3}\frac{3}{5} = \pm 4/5$. $\boxed{y = \pm 4/5}$

plug into ① to get $3 = \lambda(\pm 3/5)$ or $\boxed{\lambda = \pm 5}$

Verify $(x, y, \lambda) = (3/5, 4/5, 5)$, $(x, y, \lambda) = (-3/5, -4/5, -5)$

both satisfy ①, ②, ③. $f(3/5, 4/5) = 6\cdot\frac{3}{5} + 8\frac{4}{5} = 10$ ← MAX

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________

$f(-3/5, -4/5) = (-3 \ldots) \ldots$ ← MIN