NAME: SOLUTIONS

1 /9	2 /10	3 /8	4 /5	5 /15	6 /10	7 /8	8 /10	T /75

MATH 603 (Fall 2011) Exam 1, Oct 11

No calculators, books or notes! Show all work and give **complete explanations.**
This 75 minute exam is worth a total of 75 points.

(1) [9 pts] Let **A** be $n \times m$ and **B** be $m \times p$.
(a) Express the rows of **AB** in terms of (i) the rows of **A** and (ii) the rows of **B**.

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

So

(1) $$(AB)_{i*} = \sum_k A_{ik} B_{k*} = \overbrace{A_{i*}}^{\text{Row } i \text{ of } A} B$$

$$\vec{v}^T B = (v_1 \text{ --- } v_m) \begin{pmatrix} B_{1*} \\ \vdots \\ B_{m*} \end{pmatrix} = \sum v_k B_{k*}. \quad \text{Set } v^T = A_{i*}$$

(11) $$(AB)_{i*} = \cancel{\sum_k B_{k*} A_{k*}} = \sum_k A_{ik} \overbrace{B_{k*}}^{\text{Row } k \text{ of } B.}$$

(b) Express the matrix **AB** as a linear combination of outer products. (The outer product of two column vectors **u** and **v** is the matrix $\mathbf{uv}^T$.)

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

So $$AB = \sum_k A_{*k} B_{k*} \qquad A_{*k} \text{ is a COL}$$

$$= \sum A_{*k} (B_{*k})^T \qquad \begin{pmatrix} \vec{u} = A_{*k} \\ \vec{v} = B_{*k} \end{pmatrix}$$

(2) [10 pts] Find bases for the four fundamental subspaces of the matrix

$$\mathbf{A} = \begin{pmatrix} \boxed{2} & 3 & -2 & 6 & 3 \\ 0 & \boxed{1} & -1 & 0 & 2 \\ 0 & 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$m \times n = 4 \times 5$

$r = 3 = Rk(A)$

$\Box$ = PIVOTS.

A is already in RE form.

So $A = PU$ where $P = I_{4\times 4}$, $U = A$.

<u>R(A)</u> Basic cols in A $\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right\}$

<u>R(AT)</u> Non zero rows in $U = A$

NOTE: <u>ROW</u> VECTORS!!

$\{ (2, 3, -2, 6, 3), (0, 1, -1, 0, 2), (0, 0, 0, 3, 1) \}$

<u>N(AT)</u> Last $m - r = 1$ row(s) of P

$\{ (0, 0, 0, 1) \}$ ROW VECTOR AGAIN!

<u>N(A)</u> Back Sub with $R1 \to R1 - 3R2$, $R1 \to R1 - 2R3$ gives x_3, x_5 free.

$$\begin{pmatrix} 2 & 0 & 1 & 0 & -5 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

SOLNS OF $A\vec{x} = \vec{0}$ are

$$x_3 \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 5/2 \\ -2 \\ 0 \\ -1/3 \\ 1 \end{pmatrix} = x_3 \vec{v}_3 + x_5 \vec{v}_5.$$

<u>N(A)</u> BASIS IS $\{ \vec{v}_3, \vec{v}_5 \}$

(3) [8 pts]
(a) Does there exist a linear transformation $\mathbf{T} : \mathcal{R}^5 :\to \mathcal{R}^2$ with

$$\mathcal{N}(\mathbf{T}) = \{(x_1, x_2, x_3, x_4, x_5) \,|\, x_1 = 3x_2, x_3 = x_4 = x_5\}?$$

(b) What about if $\mathcal{N}(\mathbf{T}) = \{(x_1, x_2, x_3, x_4, x_5) \,|\, x_1 = 3x_2, x_3 = x_4\}$?

(a) N(T) is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

So $\dim N(T) = 2$

R+N THM says $\dim R(T) = 5 - \dim N(T) = 3$.

But $R(T) \subseteq \mathbb{R}^2$ $\times$!

So T DNE

(b) Define

$T(x_1, x_2, x_3, x_4, x_5) = (x_1 - 3x_2, \; x_3 - x_4)$

$T : \mathbb{R}^5 \longrightarrow \mathbb{R}^2$ is LT.

N(T) is as given!!

(4) [5 pts] Let $\mathcal{P}_2$ be the vector space of real polynomials of degree less than or equal to two. Let $\mathbf{T} : \mathcal{P}_2 \to \mathcal{R}^2$ be the linear transformation defined by $\mathbf{T}(p) = (p(0), p(1))$. Calculate the matrix $[\mathbf{T}]_{\mathcal{BC}}$ of $\mathbf{T}$ in the bases $\mathcal{B} = \{1, 1+x, 1+x+x^2\}$ for $\mathcal{P}_2$ and $\mathcal{C} = \{(1,0),(1,1)\}$ for $\mathcal{R}^2$.

$$[T]_{BC} = \left([T(u_1)]_C , [T(u_2)]_C , [T(u_3)]_C \right)$$

$$T(u_1) = T(1) = (1,1) \Rightarrow [T(u_1)]_C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T(u_2) = T(1+x) = (1,2) = -1(1,0) + 2(1,1)$$

$$\Rightarrow [T(u_2)]_C = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$T(u_3) = T(1+x+x^2) = (1,3) = -2(1,0) + 3(1,1)$$

So $[T(u_3)]_C = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

So $$[T]_{BC} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 2 & 3 \end{pmatrix}$$

(5) [15 pts] Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. A mapping $T : \mathcal{V} \to \mathcal{W}$ is called a *vector-space isomorphism* if T is one-to-one, onto, and linear.

(a) Let $\mathcal{V}$ be an n-dimensional vector space with basis $\mathcal{B}$. Define $S : \mathcal{V} \to \mathcal{R}^n$ by $S(\mathrm{v}) = [\mathrm{v}]_{\mathcal{B}}$. Prove that S is a vector-space isomorphism.

① $S(\lambda v + w) = [\lambda v + w]_{\mathcal{B}}$ $\qquad \mathcal{B} = \{u_1 - u_n\}$

$[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ | \\ c_n \end{pmatrix}$ means $v = \sum c_j u_j$

$[w]_{\mathcal{B}} = \begin{pmatrix} d_1 \\ | \\ d_n \end{pmatrix}$ $\qquad w = \sum d_j u_j$

So $\lambda v + w = \lambda \sum_j c_j u_j + \sum_j d_j u_j$

$= \sum_j (\lambda c_j + d_j) u_j$

So $\overset{S(\lambda v + w)}{=} [\lambda v + w]_{\mathcal{B}} = \begin{pmatrix} \lambda c_1 + d_1 \\ | \\ \lambda c_n + d_n \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ | \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ | \\ d_n \end{pmatrix}$

$= \lambda [v]_{\mathcal{B}} + [w]_{\mathcal{B}} = \lambda S(v) + S(w)$

② <u>1-1</u> Since S <u>linear</u> STS $S(v) = 0 \Rightarrow v = 0$.

Well $S(v) = 0 \Rightarrow [v]_{\mathcal{B}} = \vec{0} \Rightarrow \vec{v} = \sum 0 . u_j = \vec{0}$

③ onto Let $\vec{x} \in \mathbb{R}^n$ [illegible]

(b) Let $T : \mathcal{V} \to \mathcal{W}$ be a vector-space isomorphism. Prove that $\dim \mathcal{V} = \dim \mathcal{W}$.

Let $\mathcal{B} = \{u_1 \_\_\_ u_n\}$ be basis for $\mathcal{V}$.

STS $\{T(\vec{u}_1) \_\_\_ T(\vec{u}_n)\}$ is basis for $\mathcal{W}$.

<u>LI</u>

$$\vec{0} = \sum \lambda_j T(\vec{u}_j) = T\left(\sum \lambda_j \vec{u}_j\right)$$

Now T 1-1 $\Rightarrow \sum \lambda_j u_j = \vec{0}$

$\mathcal{B}$ LI $\Rightarrow \lambda_j = 0 \ \forall j$

<u>SPANNING</u> Let $\vec{w} \in \mathcal{W}$. T onto $\Rightarrow \exists \vec{v} \in V$:

$\vec{w} = T(\vec{v})$. Write $\vec{v} = \sum \lambda_j \vec{u}_j$

$\vec{w} = T\left(\sum \lambda_j u_j\right) = \sum \lambda_j T(\vec{u}_j) \in \text{Span}$ ✓

(6) [10 pts] Calculate the generalized inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 & -1 \\ 2 & 4 & 5 & -2 \\ 3 & 6 & 6 & -3 \end{pmatrix}. \qquad 3 \times 4$$

$A = BC$

$B =$ Matrix of Basic cols of A

$C =$ Non Zero Rows of RE form U of A.

GE on A gives

$$\begin{pmatrix} \boxed{1} & 2 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{So } B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \quad 3 \times 2$$

$$C = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad 2 \times 4$$

$$A^{+} \overset{\oplus}{=} \underbrace{C^{T}}_{4\times 2} \underbrace{(B^{T} A C^{T})^{-1}}_{2 \times 2} \underbrace{B^{T}}_{2\times 3} \quad \to \quad 4 \times 3.$$

Now

$$B^{T} A C^{T} = \begin{pmatrix} 84 & 32 \\ 192 & 77 \end{pmatrix}$$

Now plug into $\oplus$.

(7) [8 pts] Let **A**, **B**, and **C** be matrices such that **ABC**, **CAB**, and **BAC** are all defined. Which of the following statements are true? Justify your answers.

(a) Trace(**ABC**) = Trace(**CAB**),

(b) Trace(**ABC**) = Trace(**BAC**).

CLAIM $\quad \text{Trace}(PQ) = \text{Trace}(QP)$

PF

$$\text{Trace}(PQ) = \sum_i (PQ)_{ii} = \sum_{i,k} P_{ik} Q_{ki}$$

$$= \sum_{k,i} Q_{ki} P_{ik} = \sum_k (QP)_{kk} = \text{Trace}(QP)$$

(a) $\text{Trace}(ABC) = \text{Trace}((AB)C) = \text{Trace}(C(AB))$

$$= \text{Trace}(CAB)$$

(b) NOT TRUE.

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

$ABC = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{Trace}(ABC) = 2$

$BAC = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{Trace}(BAC) = 1$

(8) [10 pts] Let $\mathcal{D} = \{(t_1, b_1), \cdots, (t_m, b_m)\}$ be a set of $m \geq 3$ data points. By minimizing a sum of squares of errors, derive a formula for the quadratic function that is the best fit to the data, $\mathcal{D}$, in the least squares sense.

SKETCH WILL SUFFICE

$$f(t) = a + bt + ct^2$$

$$\mathcal{E}^2 = \sum_{i=1}^{m} (f(t_i) - b_i)^2$$

$$= \sum_{i=1}^{m} (a + bt_i + ct_i^2 - b_i)^2$$

$$= (A\vec{x} - b)^T (A\vec{x} - \vec{b}) = Q(\vec{x})$$

where $A = \begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} b_1 \\ | \\ b_m \end{pmatrix}$, $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Then differentiate wrt x_i as in class gives critical points satisfying $A^T A\vec{x} = A^T\vec{b}$.

So $\vec{x} = (A^TA)^{-1} A^T \vec{b}$. where A^TA is always invertible

Pledge: *I have neither given nor received aid on this exam*

Signature: ____________________