NAME: SOLUTIONS

1 /9	2 /14	3 /12	4 /16	5 /12	6 /12	T /75

MATH 603 (Fall 2011) Exam 2, Nov 8, 2011

No calculators, books or notes! Show all work and give **complete explanations**.
This 75 minute exam is worth a total of 75 points.

(1) [9 pts] Prove that $\langle \mathbf{A}|\mathbf{B}\rangle = \text{Trace}(\mathbf{A}^*\mathbf{B})$ defines an inner product on the complex vector space, $\mathcal{C}^{n\times n}$, of $n \times n$ complex matrices. [You may quote basic facts about the trace.]

1) $$\langle A|A\rangle = \text{Trace}(A^*A) = \sum_{i,j=1}^{n} (A^*)_{ij} A_{ji} = \sum_{i,j=1}^{n} \overline{A_{ji}} A_{ji}$$

$$= \sum_{i,j=1}^{n} |A_{ij}|^2 \geq 0$$

and $\langle A|A\rangle = 0 \iff A_{ij} = 0 \ \forall i,j \iff A = 0$

2) $$\langle A|\alpha B + C\rangle = \text{Trace}(A^*(\alpha B + C))$$

$$= \text{Trace}(\alpha A^*B + A^*C)$$

$$= \alpha\,\text{Trace}(A^*B) + \text{Trace}(A^*C)$$

Trace is a L.T.

$$= \alpha\langle A|B\rangle + \langle A|C\rangle$$

3) $$\overline{\langle B|A\rangle} = \overline{\text{Trace}(B^*A)} \overset{*}{=} \text{Trace}(\overline{B^*A})$$

$$= \text{Trace}(B^T\overline{A})$$

$$= \text{Trace}((B^T\overline{A})^T) \quad \text{as } \text{Trace}(M^T) = \text{Trace}(M)$$

$$= \text{Trace}(\overline{A}^T B) = \text{Trace}(A^*B) = \langle A|B\rangle$$

(2) [14 pts] Let $\mathbf{P}$ be the matrix

$$\mathbf{P} = \begin{pmatrix} 8 & 7 & 9 \\ 2 & 1 & 3 \\ 6 & 0 & 6 \end{pmatrix}.$$

(a) Calculate $\det(\mathbf{P})$ using

(i) Row operations

$$\begin{vmatrix} 8 & 7 & 9 \\ 2 & 1 & 3 \\ 6 & 0 & 6 \end{vmatrix} = - \begin{vmatrix} 2 & 1 & 3 \\ 8 & 7 & 9 \\ 6 & 0 & 6 \end{vmatrix} \qquad R2 \Leftrightarrow R1$$

$$= - \begin{vmatrix} 2 & 1 & 3 \\ 0 & 3 & -3 \\ 0 & -3 & -3 \end{vmatrix} \qquad \begin{matrix} R2 \to R2 - 4R1 \\ R3 \to R3 - 3R1 \end{matrix}$$

$$= - \begin{vmatrix} 2 & 1 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & -6 \end{vmatrix} \qquad R3 \to R3 + R2$$

$$= (-1)(2 \times 3 \times (-6)) = 36$$

(ii) A cofactor expansion Go down Column 2

$$\begin{vmatrix} 8 & 7^{-} & 9 \\ 2 & 1^{+} & 3 \\ 6 & 0^{-} & 6 \end{vmatrix} = -7 \begin{vmatrix} 2 & 3 \\ 6 & 6 \end{vmatrix} + 1 \begin{vmatrix} 8 & 9 \\ 6 & 6 \end{vmatrix} + 0$$

$$= -7(12 - 18) + 1(48 - 54)$$

$$= 36$$

(iii) Block determinants based on the blocking

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \qquad \text{where } \mathbf{A} \text{ is } 2\times 2 \text{ and } \mathbf{D} \text{ is } 1\times 1.$$

Use

$$\det(P) = \det(D)\,\det(A - B D^{-1} C) \quad \text{as } D \text{ is } 1\times 1$$

$$\left|\begin{array}{cc|c} 8 & 7 & 9 \\ 2 & 1 & 3 \\ \hline 6 & 0 & 6 \end{array}\right| = 6 \left| \begin{pmatrix} 8 & 7 \\ 2 & 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 9 \\ 3 \end{pmatrix}\begin{pmatrix} 6 & 0 \end{pmatrix} \right|$$

$$= 6 \left| \begin{pmatrix} 8 & 7 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 9 & 0 \\ 3 & 0 \end{pmatrix} \right| = 6 \begin{vmatrix} -1 & 7 \\ -1 & 1 \end{vmatrix}$$

$$= 36$$

(3) [12 pts] Let $(\mathcal{V}, \langle\cdot|\cdot\rangle_{\mathcal{V}})$ and $(\mathcal{W}, \langle\cdot|\cdot\rangle_{\mathcal{W}})$ be inner product spaces and let $\mathbf{T} : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Suppose that for each $\mathbf{w} \in \mathcal{W}$, there exists a vector $\mathbf{T}^*(\mathbf{w}) \in \mathcal{V}$ so that

$$\langle \mathbf{T}(\mathbf{v}) | \mathbf{w} \rangle_{\mathcal{W}} = \langle \mathbf{v} | \mathbf{T}^*(\mathbf{w}) \rangle_{\mathcal{V}} \qquad \forall\, \mathbf{v} \in \mathcal{V},\ \mathbf{w} \in \mathcal{W}.$$

Without using an explicit formula for $\mathbf{T}^*$, show that the mapping $\mathbf{T}^* : \mathcal{W} \to \mathcal{V}$ is linear.

NTS $\quad T^*(\alpha w_1 + w_2) = \alpha T^*(w_1) + T^*(w_2)$

Well

$$\langle v \,|\, T^*(\alpha w_1 + w_2) \rangle = \langle T(v) \,|\, \alpha w_1 + w_2 \rangle$$

$$= \alpha \langle T(v) \,|\, w_1 \rangle + \langle T(v) \,|\, w_2 \rangle$$

$$= \alpha \langle v \,|\, T^* w_1 \rangle + \langle v \,|\, T^* w_2 \rangle$$

$$= \langle v \,|\, \alpha T^* w_1 + T^* w_2 \rangle \qquad \forall v \in V.$$

So $\langle v \,|\, T^*(\alpha w_1 + w_2) - (\alpha T^* w_1 + T^* w_2) \rangle = 0 \quad \forall v \in V$

(4) [16 pts] Let $\mathbf{u}$ be a non-zero vector in C^n, and let $\mathbf{P_u}: C^n \to C^n$ be the linear transformation given by

$$\mathbf{P_u}(\mathrm{v}) = \frac{\mathbf{u}\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}}\mathrm{v}.$$

(a) Show that $\mathbf{P}_{\mathbf{u}_1}\mathbf{P}_{\mathbf{u}_2} = \mathbf{0}$ whenever $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal, and that $\mathbf{P}_{\mathbf{u}}^2 = \mathbf{P_u}$.

$$P_{u_1} P_{u_2} = \frac{u_1 u_1^*}{u_1^* u_1} \frac{u_2 u_2^*}{u_2^* u_2} = \frac{1}{\|u_1\|^2 \|u_2\|^2} u_1 \underbrace{(u_1^* u_2)}_{0 \text{ as } u_1 \perp u_2} u_2^* = 0$$

$$P_u^2 = \left(\frac{uu^*}{u^*u}\right)^2 = \frac{u\overset{1\times 1}{(u^* u)}u^*}{\|u\|^4} = \frac{\|u\|^2 . \, uu^*}{\|u\|^4}$$

$$= \frac{uu^*}{u^*u} = P_u.$$

(b) Show that $\mathbf{P}_{\mathbf{u}}^* = \mathbf{P_u}$.

$$(P_u)^* = \left(\frac{uu^*}{u^*u}\right)^* = \frac{(uu^*)^*}{\|u\|^2} = \frac{u^{**}u^*}{\|u\|^2} = \frac{uu^*}{u^*u} = P_u$$

(c) Prove that $\mathbf{P_u}$ is the orthogonal projector onto Span$\{\mathbf{u}\}$.

Let $M = \mathrm{Span}(\vec{u})$.

Since $M \oplus M^\perp = \mathbb{C}^n$, $\forall \vec{v} \in \mathbb{C}^n$ $\exists! \, \vec{x} \in M$, $\vec{y} \in M^\perp$: $\vec{v} = \vec{x} + \vec{y}$. By defn the orthogonal projector satisfies $\mathrm{PROJ}_M(\vec{v}) = \vec{x}$.

Let $\vec{v} \in V$, $\vec{x} := P_u(\vec{v})$, $\vec{y} := (I - P_u)\vec{v}$. Then $\vec{v} = \vec{x} + \vec{y}$.

STS $\vec{x} \in M$, $\vec{y} \in M^\perp$.

Well $\vec{x} = \frac{uu^*}{\quad}\vec{v} = \frac{u^*v}{\quad}$ ... (PTO)

$\vec{y} = (I - P_u)\vec{v} \in \mathcal{M}^\perp$ iff $\langle \vec{y} | \vec{u} \rangle = 0$.

Well

$$\langle \vec{y} | \vec{u} \rangle = \langle (I - P_u)\vec{v} \,/\, \vec{u} \rangle$$

$$= \langle \vec{v} | \vec{u} \rangle - \langle P_u \vec{v} \,/\, \vec{u} \rangle$$

$$= \langle \vec{v} | \vec{u} \rangle - \langle \vec{v} \,/\, P_u^* \vec{u} \rangle$$

$$= \langle \vec{v} | \vec{u} \rangle - \langle \vec{v} \,/\, P_u \vec{u} \rangle \quad \text{by } (b)$$

$$= \langle \vec{v} | \vec{u} \rangle - \langle \vec{v} \,/\, \vec{u} \rangle \quad \text{as } P_u \vec{u} = \vec{u}$$

$$= 0 \checkmark$$

** NOTE SIMPLEST TO USE (a, b, c) ABOVE FOR (d)

(d) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathcal{C}^3$. Show that the vectors

$$\mathbf{u}_1 = \mathbf{v}_1, \qquad \mathbf{u}_2 = \mathbf{v}_2 - \mathbf{P}_{\mathbf{u}_1}(\mathbf{v}_2), \qquad \mathbf{u}_3 = \mathbf{v}_3 - \mathbf{P}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathbf{P}_{\mathbf{u}_2}(\mathbf{v}_3)$$

form a basis of *orthogonal* vectors for $\mathcal{C}^3$.

STS $u_i \perp u_j$ as then normalize to get length 1 vectors and we know an ON set of 3 vectors is a basis for $\mathbb{C}^3$.

$$\langle u_2 | u_1 \rangle = \langle v_2 - P_{u_1}(v_2) \,|\, u_1 \rangle = \langle v_2 | u_1 \rangle - \langle P_{u_1}(v_2) | u_1 \rangle$$

$$= \langle v_2 | u_1 \rangle - \langle v_2 | P_{u_1} u_1 \rangle \quad \text{as } P_{u_1}^{*} = P_{u_1}$$

$$= \langle v_2 | u_1 \rangle - \langle v_2 | u_1 \rangle = 0.$$

$$\langle u_3 | u_1 \rangle = \langle v_3 - P_{u_1}(v_3) - P_{u_2}(v_3) \,|\, u_1 \rangle$$

$$= \langle v_3 | u_1 \rangle - \langle P_{u_1}(v_3) | u_1 \rangle - \langle P_{u_2}(v_3) | u_1 \rangle$$

$$= \langle v_3 | u_1 \rangle - \langle v_3 | P_{u_1} u_1 \rangle - \langle v_3 | P_{u_2} u_1 \rangle$$

$$= \langle v_3 | u_1 \rangle - \langle v_3 | u_1 \rangle - \langle v_3 | \underbrace{P_{u_2} P_{u_1} u_1}_{=0 \text{ as } u_1 \perp u_2} \rangle = 0.$$

$$\langle u_3 | u_2 \rangle = \langle v_2 - P_{u_1} v_2 \,|\, v_3 - P_{u_1} v_3 - P_{u_2} v_3 \rangle$$

$$= \langle (I - P_{u_1}) v_2 \,|\, (I - P_{u_1} - P_{u_2}) v_3 \rangle$$

$$= \langle (I - P_{u_1} - P_{u_2})(I - P_{u_1}) v_2 \,|\, v_3 \rangle$$

$$= \langle ((I - P_{u_1})^2 - P_{u_2}) v_2 \,|\, v_3 \rangle \quad \text{as } P_{u_2} P_{u_1} = 0$$

$$= \langle (I - P_{u_1} - P_{u_2}) v_2 \,|\, v_3 \rangle \quad \text{as } (I - P_u)^2 = I - P_u.$$

(5) [12 pts] Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthonormal set in an n-dimensional inner product space, $\mathcal{V}$. Prove that for any $\mathbf{x} \in \mathcal{V}$,

$$\sum_{i=1}^{k} |\xi_i|^2 \leq \|\mathbf{x}\|^2 \qquad \text{where } \xi_i = \langle \mathbf{u}_i | \mathbf{x} \rangle.$$

Hint: Consider $\|\mathbf{x} - \sum_{i=1}^{k} \xi_i \mathbf{u}_i\|^2$.

$$0 \leq \| x - \sum_{i=1}^{k} \xi_i u_i \|^2$$

$$= \langle x - \sum \xi_i u_i \,/\, x - \sum \xi_i u_i \rangle$$

$$= \|x\|^2 - \langle \sum \xi_i u_i | x \rangle - \langle x / \sum \xi_i u_i \rangle$$

$$+ \langle \sum \xi_i u_i | \sum \xi_j u_j \rangle$$

$$= \|x\|^2 - \sum_i \overline{\xi_i} \langle u_i | x \rangle - \sum_i \xi_i \langle x / u_i \rangle$$

$$+ \sum_{ij} \overline{\xi_i} \xi_j \langle u_i / u_j \rangle$$

$$= \|x\|^2 - \sum_i \overline{\xi_i} \xi_i - \sum_i \xi_i \overline{\xi_i} + \sum_i \overline{\xi_i} \xi_i$$

as $\langle u_i / u_j \rangle = \delta_{ij}$

$$= \|x\|^2 - \sum |\xi_i|^2.$$

So $\quad \sum |\xi_i|^2 \leq \|x\|^2.$

(6) [12 pts] Let $\mathbf{A}$ be an $n \times n$ orthogonal matrix and let $\mathbf{E}_{ij}$ be the $n \times n$ matrix whose (i,j)-entry is equal to 1 and all other entries are zero. Prove that

METHOD 1

$$\frac{d}{dt}\det(\mathbf{A} + t\mathbf{E}_{ij}) = \mathbf{A}_{ij}\det(\mathbf{A}).$$

$$\det(A + tE_{ij}) = \det(A + t\vec{e}_i\vec{e}_j^{\,T})$$

$$= \det(A)\left(1 + \vec{e}_j^{\,T} A^{-1} t\vec{e}_i\right)$$

$$= \det(A)\left(1 + t\vec{e}_j^{\,T} A^T \vec{e}_i\right) \qquad A^T = A^{-1}$$

$$= \det(A)\left(1 + t\,A^T_{ji}\right) = \det(A)\left(1 + tA_{ij}\right)$$

So taking d/dt gives result.

METHOD 2

$$\det(A + tE_{ij}) = \det\begin{bmatrix} A_{1*} \\ \vdots \\ A_{i*} + t\vec{e}_j^{\,T} \\ \vdots \\ A_{n*} \end{bmatrix} \overset{\text{LIN IN ROW } i}{=} \det(A) + t\det\begin{bmatrix} A_{1*} \\ | \\ e_j^T \\ | \\ A_{n*} \end{bmatrix} \leftarrow \text{ROW } i$$

$$\text{So } \frac{d}{dt}\det(A + tE_{ij}) = \det\begin{bmatrix} A_{1*} \\ | \\ e_j^T \\ | \\ A_{n*} \end{bmatrix} = \det\left(A^T_{*1} \text{ — } e_j \text{ — } A^T_{*n}\right)$$

$\boxed{\det(Q) = \det(Q^T)}$

NOW $A^T x = e_j$ has solution $x = Ae_j$, ~~$x_i =$~~

$x = A_{*j}$, $x_i = A_{ij}$

But by Cramer's Rule
$$x_i = \frac{\det\left(A^T_{*1} \text{ — } e_j \text{ — } A^T_{*n}\right)}{\det(A)}$$

Pledge: *I have neither given nor received aid on this exam*

Signature: ______________________________

So

$$\frac{d}{dt}\det(A + tE_{ij}) = x_i\det(A) = A_{ij}\det A$$